On Statistical Structures and Dual Connections on the Subbundle HM of Finsler Spaces

by Tetsuya NAGANO*

Abstract

The author studies the statistical structures and the dual connections on the subbundle HM of TM on Finsler spaces. In particular, the case given two Finsler connections ∇, ∇^* that have the different non-linear connections N, N^* is treated in this paper.
Keywords: statistical structure, dual connection, Finsler metric, Finsler connection, non-linear connection.

Introduction

The author investigated the statistical structures on TM of Finsler spaces([N05],[N06-1]). Further the author studied the dual connections of Finsler connections on TM([N06-2]). Finsler Geometry is treated as the geometry on the subbundle VM of TM in [A-K06]. The author and Prof.Aikou had the lecture "Finsler manifolds satisfying $R_D^{HH} = 0$" at Workshop on Finsler Geometry and its Applications in May 29-June 2, 2007, Hungary. In that lecture, Finsler geometry was treated as the geometry on the subbundle HM of TM on Finsler spaces and the notion of the dual connection of Finsler connection was stated as the structure between two Finsler connections with the same non-linear connection.

Now, according to the lecture and [A-K06], the author studies the statistical structures and the dual connections on the subbundle HM of TM on Finsler spaces because HM is isomorphism to VM. In this paper, the author, however, treats the two Finsler connections with the different non-linear connections.

In §1, general preparations and the notion of the statistical structure and the dual connections are stated. In §2, the notion of Finsler spaces and Finsler connections are stated. In §3, the torsions of Finsler connections is shown. In §4, the notion of the statistical structures and dual connections on HM is studied and the conditions that two Finsler connections with the different non-linear connections is shown. In addition, its application and an open problem (in spite of the similar problem is still solved in Riemannian geometry) is stated.

In this paper, the author refers [A-N00] for the notion of statistical structures and dual connections and obeys [M86] with respect to the notations of the covariant derivation and the position of its indices. In addition the author obeys [A-K06] with respect to the global notations of Finsler geometry.

*Department of Info-media, Siebold University of Nagasaki. E-mail : hnagano@sun.ac.jp

1 Statistical Structures and Dual Connections

We state the notion of the statistical structure and the dual connection in Riemannian case.

Definition 1.1 *Let M be an $n-$dimensional differentiable manifold, h a semi-Riemannian metric, and ∇ a symmetric(torsion free) linear connection on M. If ∇h is totally symmetric, then the pair (h, ∇) is called a statistical structure on M and (M, h, ∇) is called a statistical manifold.*

On a statistical manifold (M, h, ∇), the symmetric tensor field $K := \nabla h$ is called the *cubic form* and written in the local coordinate system (x^i) of M as follows:

$$K = (K_{ijk}) = K_{ijk} dx^i \otimes dx^j \otimes dx^k, \tag{1.1}$$

where $K_{ijk} := h_{ij,k}$ and $,k$ is covariant derivative of $\frac{\partial}{\partial x^k}$.
In addition, the tensor field L defined by

$$h(L(X, Y), Z) = K(X, Y, Z) = (\nabla_Z h)(X, Y) \tag{1.2}$$

is called the *skewness operator* of (M, h, ∇) and we put L_{ij}^k to the coefficient of L, where $L_{ij}^k = K_{irj} h^{rk}$ and $(h^{ij}) = (h_{ij})^{-1}$.
Furthermore, we put ∇^0 to the Levi-Civita connection of h, then

$$\nabla = \nabla^0 - \frac{1}{2} L \quad (i.e.\ \Gamma_{ij}^k = \Gamma_{ij}^{0k} - \frac{1}{2} L_{ij}^k) \tag{1.3}$$

is satisfied, where Γ_{ij}^k and Γ_{ij}^{0k} are the coefficients of ∇ and ∇^0, respectively.

Definition 1.2 *Let h be a semi-Riemannian metric and ∇ a symmetric linear connection. If another symmetric linear connection ∇^* satisfies the following relation*

$$Zh(X, Y) = h(\nabla_Z X, Y) + h(X, \nabla_Z^* Y), \tag{1.4}$$

then ∇^* is called the dual connection of ∇ with respect to h, where X, Y, Z are any vector fields.

Remark 1.1 *In general, for every statistical manifold (M, h, ∇), there exists a naturally symmetric trilinear form K called the cubic form. Conversely, let (M, h, K) be a semi-Riemannian manifold with a symmetric trilinear form K. If we define the tensor field L of type (1,2) by $h(L(X, Y), Z) := K(X, Y, Z)$, and an affine connection ∇ by $\nabla := \nabla^0 - \frac{1}{2} L$. Then ∇ is symmetric and satisfies $\nabla h = K$. Therefore the triplet (M, h, ∇) becomes a statistical manifold. Thus to equip a statistical structure (h, ∇) is equivalent to equip a pair (h, K) consisting of a semi-Riemannian metric h and a symmetric trilinear form K. Hence (h, K) is also called a statistical structure. Further, if (M, h, ∇) is a statistical manifold, then (M, h, ∇^*) is also because of the relation $\nabla^0 = \dfrac{1}{2}(\nabla + \nabla^*)$.*

2 Finsler Spaces and Finsler Connections

Let M be an $n-$dimensional differentiable manifold and let $\pi : TM \to M$ be the tangent bundle of M. We denote by $v = (x, y)$ the points in TM if $x \in M$ and $y \in \pi^{-1}(x) = T_xM$. We denote by $z(M)$ the zero section of TM, and by $TM^\times$ the split tangent bundle $TM\backslash z(M)$. Further, let $U \subset M$ be an open set with a local coordinate $(x^1, \cdots, x^n)$. By setting $v = \sum y^i(\partial/\partial x^i)_x$ for every $v \in \pi^{-1}(U)$, we can introduce a local coordinate $(x^i, y^i) = (x^1, \cdots, x^n, y^1, \cdots, y^i)$ on $\pi^{-1}(U)$. Thus we have the local coordinate system $\{\pi^{-1}(U), (x^i, y^i)\}$ on TM.

Definition 2.1 *A function $F : TM \to \mathbb{R}$ is called a Finsler metric on M if*

1. $F(x, y) \geq 0$, *and* $F(x, y) = 0$ *if and only if* $y = 0$,
2. $F(x, \lambda y) = \lambda F(x, y)$ *for* $\forall\lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$,
3. $F(x, y)$ *is differentiable on* $TM^\times$
4. *the Hessian* (g_{ij}) *defined by*

$$g_{ij}(x, y) = \frac{\partial^2(\frac{1}{2}F^2)}{\partial y^i \partial y^j} \tag{2.1}$$

is regular.

Then the pair (M, F) is called a Finsler space(or Finsler manifold). For each $X \in T_xM$, its norm $||X||$ is defined by $||X|| = F(x, X)$.

A Finsler metric F is said to be *convex* if $F^2/2$ is *strictly convex* on each tangent space T_xM, that is, the Hessian (g_{ij}) is positive-definite. The convexity of F is equivalent to the one of the unit ball $B_x = \{y \in T_xM | F(x, y) \leq 1\}$. In addition, we also call g_{ij} Finsler metric.

Next, we will state Finsler connections. So we take the following notations about the indices $i, j, k, \cdots, A, B, C, \cdots, \bar{1}, \bar{2}, \bar{3}, \cdots$.

Let $i, j, k, \cdots = 1, 2, \cdots, n;\ A, B, C, \cdots = 1, 2, \cdots, n, n+1, \cdots, 2n$, be the ranges of indices $i, j, k, \cdots,\ A, B, C, \cdots$, respectively and $\bar{1}, \bar{2}, \cdots, \bar{n}$ are meant by $\bar{1} = n+1, \bar{2} = n+2, \cdots, \bar{n} = 2n$.

Definition 2.2 *Let $N = (N^i_j)$ be an nonlinear connection and ∇ a linear connection on $TM^\times$, respectively. If, for the adapted frames $(\delta_i, \partial_{\bar{i}}) = (\frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r}, \frac{\partial}{\partial y^i})$ as frames of $TM^\times$ on each local coordinate (x^i, y^i), ∇ satisfies the following formulae:*

$$\nabla_{\delta_j}\delta_i = F^r_{ij}\delta_r, \quad \nabla_{\delta_j}\partial_{\bar{i}} = F^r_{ij}\partial_{\bar{r}}, \quad \nabla_{\partial_{\bar{j}}}\delta_i = C^r_{ij}\delta_r, \quad \nabla_{\partial_{\bar{j}}}\partial_{\bar{i}} = C^r_{ij}\partial_{\bar{r}}, \tag{2.2}$$

then the pair $F\Gamma = (N, \nabla)$ is called a Finsler connection on M and $(N^i_j, F^i_{jk}, C^i_{jk})$ is called the coefficients of the Finsler connection $F\Gamma = (N, \nabla)$. In addition, we also call $(N^i_j, F^i_{jk}, C^i_{jk})$ Finsler connection and we denote the Finsler connection by $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ or $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$.

Further, for the coframes $(dx^i, \delta y^i) = (dx^i, dy^i + N^i_j dx^j)$ of $(\delta_i, \partial_{\bar{i}})$, the Finsler connection $F\Gamma$ satisfies

$$(2.3) \quad \nabla_{\delta_j} dx^i = -F^i_{rj} dx^r,\ \nabla_{\delta_j} \delta y^i = -F^i_{rj} \delta y^r,\ \nabla_{\partial_{\bar{j}}} dx^i = -C^i_{rj} dx^r,\ \nabla_{\partial_{\bar{j}}} \delta y^i = -C^i_{rj} \delta y^r.$$

3 Torsions of Finsler connections

We will write the coefficients of torsion tensor field $\bar{T}$ of ∇ with respect to the adapted frames $\partial_A = (\delta_i, \partial_{\bar{i}})$ by $\bar{T}^A_{BC}$. Since the following formulae of the torsion tensor field of ∇

$$(3.1) \qquad \bar{T}(\partial_B, \partial_C) = \nabla_{\partial_B} \partial_C - \nabla_{\partial_C} \partial_B - [\partial_B, \partial_C] = \bar{T}^A_{BC} \partial_A,$$

we have

$$(3.2) \qquad \begin{aligned} &\bar{T}(\delta_i, \delta_j) = \bar{T}^r_{ij} \delta_r + \bar{T}^{\bar{r}}_{ij} \partial_{\bar{r}},\ \bar{T}(\delta_i, \partial_{\bar{j}}) = \bar{T}^r_{i\bar{j}} \delta_r + \bar{T}^{\bar{r}}_{i\bar{j}} \partial_{\bar{r}}, \\ &\bar{T}(\partial_{\bar{i}}, \delta_j) = \bar{T}^r_{\bar{i}j} \delta_r + \bar{T}^{\bar{r}}_{\bar{i}j} \partial_{\bar{r}},\ \bar{T}(\partial_{\bar{i}}, \partial_{\bar{j}}) = \bar{T}^r_{\bar{i}\bar{j}} \delta_r + \bar{T}^{\bar{r}}_{\bar{i}\bar{j}} \partial_{\bar{r}}, \end{aligned}$$

where

$$(3.3) \qquad \begin{aligned} &\bar{T}^r_{ij} = F^r_{ji} - F^r_{ij},\ \bar{T}^{\bar{i}}_{jk} = -\delta_k N^i_j + \delta_j N^i_k,\ \bar{T}^i_{j\bar{k}} = -C^i_{jk},\ \bar{T}^{\bar{i}}_{j\bar{k}} = F^i_{kj} - \partial_{\bar{k}} N^i_j, \\ &\bar{T}^i_{\bar{j}k} = C^i_{kj},\ \bar{T}^{\bar{i}}_{\bar{j}k} = \partial_{\bar{j}} N^i_k - F^i_{jk},\ \bar{T}^i_{\bar{j}\bar{k}} = 0,\ \bar{T}^{\bar{i}}_{\bar{j}\bar{k}} = -C^i_{jk} + C^i_{kj}. \end{aligned}$$

If we rewrite the relation (3.2) by Finsler torsion tensors T, R, P, S, C of Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$, we have

$$(3.4) \qquad \begin{aligned} &\bar{T}(\delta_i, \delta_j) = -T^r_{ij} \delta_r - R^r_{ij} \partial_{\bar{r}},\ \bar{T}(\delta_i, \partial_{\bar{j}}) = -C^r_{ij} \delta_r - P^r_{ij} \partial_{\bar{r}}, \\ &\bar{T}(\partial_{\bar{i}}, \delta_j) = C^r_{ji} \delta_r + P^r_{ji} \partial_{\bar{r}},\ \bar{T}(\partial_{\bar{i}}, \partial_{\bar{j}}) = -S^r_{ij} \partial_{\bar{r}}, \end{aligned}$$

where $T^i_{jk} = F^i_{jk} - F^i_{kj},\ R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k,\ P^i_{jk} = \partial_{\bar{k}} N^i_j - F^i_{kj},\ S^i_{jk} = C^i_{jk} - C^i_{kj}$.

If we assume that the Finsler connection ∇ is symmetric, namely, torsion free as a linear connection of $TM^\times$, then all Finsler torsion tensors vanish. This is inconvenient for Finsler geometry(Theorem3.2,4.2,5.2 in [N05]). In the Riemannian geometry, however, the symmetric property of the connection is necessary in the notion of the dual connection of ∇. By the detail investigation for the *dual Finsler connection*, we will notice that it is sufficiently that the symmetric property of Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ is $T^i_{jk} = 0$ and $S^i_{jk} = 0$ only. Therefore we present the notion of the statistical structure for Finsler spaces as the following section.

4 Statistical Structures and Dual Connections on HM

Now, Finsler geometry is developed globally as the geometry of the vertical subbundle of the tangent bundle in [A-K06]. If the nonlinear connection is given, the horizontal subspace of the tangent space of TM is equivalent to the vertical subspace of the tangent space of TM at every point on TM. Namely, we put HM and VM to the horizontal subbundle and the vertical subbundle, respectively, then the following decomposition

$$TTM = HM \oplus VM \tag{4.1}$$

is satisfied, where HM is locally spanned by $\{\delta_i\}$ and VM is locally spanned by $\{\partial_{\bar{i}}\}$ and the map τ

$$\tau : HM \to VM \mid \delta_i \to \partial_{\bar{i}} \tag{4.2}$$

is homomorphism.

According to the decomposition (5.1), we can express any vector field X on TM as $X = X^H + X^V$, where $X^H \in HM$ and $X^V \in VM$, and the exterior differential d also split as $d = d^H + d^V$, where d^H is the differential along HM and d^V is the one along VM. In addition the dual bundle HM^* and VM^* are locally spanned by $\{dx^i\}$ and $\{\delta y^i\}$, respectively.

Furthermore, the Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ also split as

$$\nabla = \nabla^H + \nabla^V. \tag{4.3}$$

Now *we consider the geometry on the horizontal subbundle HM*, namely, the Finsler metric $g = (g_{ij})$ is regarded a metric on the horizontal space as follows

$$g = g_{ij} dx^i \otimes dx^j \tag{4.4}$$

at each point on $TM^\times$.

Firstly, we study the statistical structure on HM, namely, for the Finsler connection $F\Gamma = (N, \nabla)$ and any horizontal vector fields X, Y, Z on $TM^\times$, we consider the case that $\nabla^H g$ is totally symmetric with respect to the metric (4.4).

Let (M, F) be a Finsler space with a Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfying $T^i_{jk} = 0$. Because of the definition, we have on (x^i, y^i)

$$\nabla^H g = g_{ij|k} dx^i \otimes dx^j \otimes dx^k. \tag{4.5}$$

Therefore, for any horizontal vector fields $X = X^i \delta_i, Y = Y^i \delta_i, Z = Z^i \delta_i$ on $TM^\times$

$$(\nabla^H_Z g)(X, Y) = g_{ij|k} X^i Y^j Z^k \tag{4.6}$$

is satisfied.

Thus we have

Theorem 4.1 *Let (M, F) be a Finsler space with a Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfying $T^i_{jk} = 0$. $\nabla^H g$ is totally symmetric with respect to the metric* (4.4) *if and only if $g_{ij|k}$ is totally symmetric.*

Here we put the definition as follows

Definition 4.1 *Let (M,F) be a Finsler space with a Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfying $T^i_{jk} = 0$. If $\nabla^H g$ is totally symmetric with respect to the metric (4.4), then (F,∇) is called the statistical structure on the subbundle HM of the Finsler space (M,F).*

Then we can proof the following theorem

Theorem 4.2 *There exist statistical structures on HM of a Finsler space (M,F).*

Proof
We can take the following Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$,

$$N^i_j = \gamma^i_{0j} - \frac{1}{2} g^{il} \partial_{\bar{r}} g_{lj} \gamma^r_{00}, \tag{4.7}$$

$$F^i_{jk} = \frac{1}{2} g^{il} (\delta_j g_{lk} + \delta_k g_{jl} - \delta_l g_{jk}) - \frac{1}{2} g^{il} \partial_{\bar{k}} g_{lj}, \tag{4.8}$$

$$C^i_{jk} = 0, \tag{4.9}$$

where

$$\gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}). \tag{4.10}$$

The relation $F^i_{jk} = F^i_{kj}$ follows from $g_{ij} = g_{ji}$ and $\partial_{\bar{k}} g_{ij} = \partial_{\bar{j}} g_{ki} = \partial_{\bar{i}} g_{jk}$. So $T^i_{jk} = 0$ is satisfied.

$$\begin{aligned} g_{ij|k} &= \delta_k g_{ij} - F^r_{ik} g_{rj} - F^r_{jk} g_{il} \\ &= \frac{1}{2} g^{rm} (\partial_{\bar{k}} g_{mi}) g_{rj} + \frac{1}{2} g^{rm} (\partial_{\bar{k}} g_{mj}) g_{ir} \\ &= \partial_{\bar{k}} g_{ij} = \frac{1}{2} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}. \end{aligned} \tag{4.11}$$

Thus the symmetric property of $g_{ij|k}$ is satisfied. Therefore (F,∇) is a statistical structure.

Q.E.D.

Remark 4.1 *In the above theorem, if we put $\bar{F}^i_{jk} = \frac{1}{2} g^{il}(\delta_j g_{lk} + \delta_k g_{jl} - \delta_l g_{jk})$, the connection $(N^i_j, \bar{F}^i_{jk}, C^i_{jk})$ is so-called "Rund connection(or Chern connection)" and the Finsler tensor field $g^{il}\partial_{\bar{k}} g_{lj}$ is so-called "Cartan tensor C".*

Next, we study dual connections of Finsler connections $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$. We set the following situation, firstly.

A Finsler space (M,F) and a Finsler connection $F\Gamma = (N,\nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfying $T^i_{jk} = 0$ are given. Further another Finsler connection $F\Gamma^* = (N^*, \nabla^*) = ({N^*}^i_j, {F^*}^i_{jk}, {C^*}^i_{jk})$ satisfying ${T^*}^i_{jk} = 0$ are given. In this case we have two frames $(\delta_i, \partial_{\bar{i}}) = (\partial_i - N^r_i \partial_{\bar{r}}, \partial_{\bar{i}})$ and $(\delta^*_i, \partial_{\bar{i}}) = (\partial_i - {N^*}^r_i \partial_{\bar{r}}, \partial_{\bar{i}})$ of $TM^\times$, and two coframes $(dx^i, \delta y^i) =$

$(dx^i, dy^i + N^i_j dx^j)$ and $(dx^i, \delta^* y^i) = (dx^i, dy^i + N^{*i}_{\ j} dx^j)$, respectively.
We put

$$N^{*i}_{\ j} - N^i_j = B^i_j. \tag{4.12}$$

Then we have

$$\delta^*_i = \delta_i - B^r_i \partial_{\bar r}, \qquad \delta^* y^i = \delta y^i + B^i_r dx^r. \tag{4.13}$$

Further we have

$$\nabla_{\delta_j} \delta_i = F^r_{ij} \delta_r, \quad \nabla_{\delta_j} \partial_{\bar i} = F^r_{ij} \partial_{\bar r}, \quad \nabla_{\partial_{\bar j}} \delta_i = C^r_{ij} \delta_r, \quad \nabla_{\partial_{\bar j}} \partial_{\bar i} = C^r_{ij} \partial_{\bar r}, \tag{4.14}$$

$$\nabla_{\delta_j} dx^i = -F^i_{rj} dx^r, \ \nabla_{\delta_j} \delta y^i = -F^i_{rj} \delta y^r, \ \nabla_{\partial_{\bar j}} dx^i = -C^i_{rj} dx^r, \ \nabla_{\partial_{\bar j}} \delta y^i = -C^i_{rj} \delta y^r \tag{4.15}$$

and

$$\nabla^*_{\delta^*_j} \delta^*_i = F^{*r}_{\ ij} \delta^*_r, \quad \nabla^*_{\delta^*_j} \partial_{\bar i} = F^{*r}_{\ ij} \partial_{\bar r}, \quad \nabla^*_{\partial_{\bar j}} \delta^*_i = C^{*r}_{\ ij} \delta^*_r, \quad \nabla^*_{\partial_{\bar j}} \partial_{\bar i} = C^{*r}_{\ ij} \partial_{\bar r}, \tag{4.16}$$

(4.17)
$$\nabla^*_{\delta^*_j} dx^i = -F^{*i}_{\ rj} dx^r, \ \nabla^*_{\delta^*_j} \delta^* y^i = -F^{*i}_{\ rj} \delta^* y^r, \ \nabla^*_{\partial_{\bar j}} dx^i = -C^{*i}_{\ rj} dx^r, \ \nabla^*_{\partial_{\bar j}} \delta^* y^i = -C^{*i}_{\ rj} \delta^* y^r.$$

From (4.13) and (4.16) we also have

$$\begin{aligned} &\nabla^*_{\delta_j} \delta_i = (F^{*p}_{\ ij} + C^{*p}_{\ ir} B^r_j) \delta_p + (B^p_{i|^* j} + B^p_i |^*_r B^r_j) \partial_{\bar p}, \\ &\nabla^*_{\delta_j} \partial_{\bar i} = (F^{*p}_{\ ij} + C^{*p}_{\ ir} B^r_j) \partial_{\bar p}, \quad \nabla^*_{\partial_{\bar j}} \delta_i = C^{*p}_{\ ij} \delta_p + B^p_i |^*_j \partial_{\bar p}, \quad \nabla^*_{\partial_{\bar j}} \partial_{\bar i} = C^{*p}_{\ ij} \partial_{\bar p}. \end{aligned} \tag{4.18}$$

In addition, from (4.18) we have for any horizontal vector fields $X = X^i \delta_i$, $Y = Y^i \delta_i$ and $Z = Z^i \delta_i$ on $TM^\times$

$$\nabla^H_Z X = Z^r X^i_{|r} \delta_i, \ \nabla^{*H}_{\ Z} Y = Z^r (Y^i_{|^* r} + B^l_r Y^i |^*_l) \delta_i. \tag{4.19}$$

According to our standpoints, we should consider the case that, for the metric (4.4) and any horizontal vector fields $X = X^i \delta_i$, $Y = Y^i \delta_i$ and $Z = Z^i \delta_i$ on $TM^\times$,

$$Zg(X, Y) = g(\nabla^H_Z X, Y) + g(X, \nabla^{*H}_{\ Z} Y). \tag{4.20}$$

From (4.4), the left hand side of (4.20) is

$$\begin{aligned} Zg(X, Y) &= Z^r \delta_r (g_{ij} X^i Y^j) \\ &= Z^r (\delta_r g_{ij} X^i Y^j + g_{ij} \delta_r X^i Y^j + g_{ij} X^i \delta_r Y^j). \end{aligned} \tag{4.21}$$

On the other hand, from the first equation of (4.19), the first term of the right hand side of (4.20) is

$$\begin{aligned} g(\nabla^H_Z X, Y) &= g_{ij} dx^i (Z^r X^p_{|r} \delta_p) dx^j (Y^r \delta_r) \\ &= g_{ij} X^i_{|r} Y^j Z^r \end{aligned} \tag{4.22}$$

and from the second equation of (4.19), the second term of the right hand side of (4.20) is

$$g(X, \nabla^{*H}_{Z}Y) = g_{ij}dx^i(X^r\delta_r)dx^j(Z^r(Y^p_{|*r} + B^l_rY^p|^*_l)\delta_p)$$
$$= g_{ij}X^i(Y^j_{|*r} + B^l_rY^j|^*_l)Z^r. \tag{4.23}$$

Therefore, from (4.22) and (4.23), the right hand side of (4.20) satisfies as follows

$$g(\nabla^H_Z X, Y) + g(X, \nabla^{*H}_{Z}Y) = g_{ij}X^i_{|r}Y^jZ^r + g_{ij}X^i(Y^j_{|*r} + B^l_rY^j|^*_l)Z^r. \tag{4.24}$$

Finally, from (4.21), (4.24) and the arbitrariness of Z, we have

$$\delta_r g_{ij}X^iY^j + g_{ij}\delta_r X^iY^j + g_{ij}X^i\delta_r Y^j = g_{ij}X^i_{|r}Y^j + g_{ij}X^iY^j_{|*r} + g_{ij}X^iB^l_rY^j|^*_l. \tag{4.25}$$

Here we take the following relations

$$Y^j_{|*r} \doteq \delta^*_r Y^j + F^{*j}_{pr}Y^p = \delta_r Y^j - B^l_r\partial_{\bar{l}}Y^j + F^{*j}_{pr}Y^p \tag{4.26}$$

and notice $X^i_{|r} = \delta_r X^i + F^i_{pr}X^p$ and $Y^j|^*_l = \partial_{\bar{l}}Y^j + C^{*j}_{pl}Y^p$, then from (4.25) we have

$$\delta_r g_{ij}X^iY^j = (g_{kj}F^k_{ir} + g_{ik}F^{*k}_{jr} + g_{ik}C^{*k}_{jl}B^l_r)X^iY^j. \tag{4.27}$$

Since the arbitrariness of $X$and Y, we can obtain

$$\delta_r g_{ij} = g_{kj}F^k_{ir} + g_{ik}F^{*k}_{jr} + g_{ik}C^{*k}_{jl}B^l_r. \tag{4.28}$$

Thus we have

Theorem 4.3 *Let (M, F) be a Finsler space and $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfying $T^i_{jk} = 0$ a Finsler connection. Further,let $\nabla^* = (N^{*i}_j, F^{*i}_{jk}, C^{*i}_{jk})$ be another Finsler connection satisfying $T^{*i}_{jk} = 0$. For the metric (4.4) on HM and any horizontal vector fields $X = X^i\delta_i$, $Y = Y^i\delta_i$ and $Z = Z^i\delta_i$ on $TM^\times$, the relation (4.20) is satisfied if and only if the relation (4.28) is satisfied.*

On the other hand, we consider the case for the metric $g = g_{ij}dx^i \otimes dx^j$ on another subbundle H^*M spanned by $\{\delta^*_i\}$ and for any horizontal vector fields $X^* = X^{*i}\delta^*_i, Y^* = Y^{*i}\delta^*_i$ and $Z = Z^{*i}\delta^*_i$ on $TM^\times$

$$Z^*g(X^*, Y^*) = g(\nabla^{*H}_{Z^*}X^*, Y^*) + g(X^*, \nabla^H_{Z^*}Y^*). \tag{4.29}$$

If we apply the theorem 4.3 to H^*M and horizontal vector fields $X^* = X^{*i}\delta^*_i, Y^* = Y^{*i}\delta^*_i$ and $Z = Z^{*i}\delta^*_i$, we obtain the relation as follows:

$$\delta^*_r g_{ij} = g_{kj}F^{*k}_{ir} + g_{ik}F^k_{jr} - g_{ik}C^k_{jl}B^l_r. \tag{4.30}$$

From (4.13), the relation $\delta^*_r g_{ij} = \delta_r g_{ij} - B^l_r\partial_{\bar{l}}g_{ij}$ hold. Therefore we have

$$\delta_r g_{ij} = g_{kj}F^{*k}_{ir} + g_{ik}F^k_{jr} + (\partial_{\bar{l}}g_{ij} - g_{ik}C^k_{jl})B^l_r. \tag{4.31}$$

On the other hand, the relation $g_{ji} = g_{ij}$ hold, so

$$\begin{aligned} \delta_r g_{ij} &= g_{ki} F^{*k}_{\ jr} + g_{jk} F^k_{ir} + (\partial_{\bar{l}} g_{ji} - g_{jk} C^k_{il}) B^l_r \\ &= g_{ik} F^{*k}_{\ jr} + g_{kj} F^k_{ir} + (\partial_{\bar{l}} g_{ij} - g_{kj} C^k_{il}) B^l_r \end{aligned} \tag{4.32}$$

is satisfied. From (4.28) and (4.32), we obtain

$$\partial_{\bar{l}} g_{ij} B^l_r = g_{kj} C^k_{il} B^l_r + g_{ik} C^{*k}_{\ jl} B^l_r. \tag{4.33}$$

Therefore we have the following theorem:

Theorem 4.4 *Let (M, F) be a Finsler space and $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfying $T^i_{jk} = 0$ a Finsler connection. Further,let $\nabla^* = (N^{*i}_{\ j}, F^{*i}_{\ jk}, C^{*i}_{\ jk})$ be another Finsler connection satisfying $T^{*i}_{\ jk} = 0$. The relations (4.20) and (4.29) are satisfied if and only if the relation (4.28) and (4.33) are satisfied.*

Here we put the definition as follows

Definition 4.2 *Let (M, F) be a Finsler space, and let $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ be a Finsler connection satisfying $T^i_{jk} = 0$ and $\nabla^* = (N^{*i}_{\ j}, F^{*i}_{\ jk}, C^{*i}_{\ jk})$ another Finsler connection satisfying $T^{*i}_{\ jk} = 0$. If ∇, ∇^* satisfy the relations (4.20) and (4.29), then ∇, ∇^* are called the h-dual connection.*

Next, we study the case:
"If (F, ∇) is a statistical structure on HM and ∇, ∇^* are h-dual connections, then is (F, ∇^*) a statistical structure on H^*M?"
This is an open problem, yet. We point out the assumptions and the conclusion. Since (F, ∇) is a statistical structure on HM, we have the assumption that $g_{ij|k}$ is totally symmetric with respect to the indices i, j and k. Further, since ∇, ∇^* are h-dual connections, we have other assumptions (4.28) and (4.33). Then the conclusion that we must indicate is the totally symmetric property of $g_{ij|^*k}$ with respect to the indices i, j and k.
We have only the following relations because $T^i_{jk} = 0$, $T^{*i}_{\ jk} = 0$ and $g_{ij|k}$ is totally symmetric:

$$C^{*i}_{\ jl} B^l_k = C^{*i}_{\ kl} B^l_j, \qquad g_{ij}|_l B^l_r + g_{ik} C^k_{jl} B^l_r = g_{ir}|_l B^l_j + g_{ik} C^k_{rl} B^l_j \tag{4.34}$$

and

$$g_{ij|^*r} + g_{ik} C^k_{jl} B^l_r = g_{ir|^*j} + g_{ik} C^k_{rl} B^l_j. \tag{4.35}$$

If we can proof the relation $C^i_{jl} B^l_k = C^i_{kl} B^l_j$, then we have $g_{ij|^*k} = g_{ik|^*j}$, namely, $g_{ij|k}$ is totally symmetric. We, however, do not reach the conclusion, yet.

Now, we can construct another Finsler connection from the given Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ as follows:

$$\tilde{N}^i_j = F^i_{0j}, \ \tilde{F}^i_{jk} = F^i_{jk} - C^i_{jr} D^r_k, \ \tilde{C}^i_{jk} = C^i_{jk}, \ D^i_j = F^i_{0j} - N^i_j, \tag{4.36}$$

then we obtain the new Finsler connection $\tilde{\nabla} = (\tilde{N}^i_j, \tilde{F}^i_{jk}, \tilde{C}^i_{jk})$ and D^i_j is called the coefficient of the deflection tensor field.

Lastly, we consider the case that ∇ and $\tilde{\nabla}$ are h-dual connections.
We set $N^* = \tilde{N},\ F^* = \tilde{F},\ C^* = \tilde{C},\ B = D$, then from (4.28)

$$(4.37)\qquad \begin{aligned}\delta_r g_{ij} &= g_{kj}F^k_{ir} + g_{ik}(F^k_{jr} - C^k_{jl}D^l_r) + g_{ik}C^k_{jl}D^l_r \\ &= g_{kj}F^k_{ir} + g_{ik}F^k_{jr}\end{aligned}$$

is satisfied. Namely, we have "h-metrical"

$$(4.38)\qquad g_{ij|r} = 0.$$

By the same way, we have

$$(4.39)\qquad g_{ij|^*k} = 0.$$

In addition, from (4.33), we have $\partial_{\bar{l}} g_{ij} D^l_r = g_{kj}C^k_{il}D^l_r + g_{ik}C^k_{jl}D^l_r$, that is,

$$(4.40)\qquad (\partial_{\bar{l}} g_{ij} - g_{kj}C^k_{il} - g_{ik}C^k_{jl})D^l_r = 0$$

is satisfied. Namely, we have

$$(4.41)\qquad g_{ij}|_l D^l_r = 0.$$

Theorem 4.5 *Let $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ be a given Finsler connection. If ∇ and $\tilde{\nabla}$ are h-dual connection, then we have* (4.38), (4.39) *and* (4.41).

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