On Statistical Structures and the lifts of Finsler Metrics

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Abstract

There is not the notion of its statistical structure for Finsler spaces yet. The author lifts the Finsler metric to its tangent bundle and studies the conditions that its tangent bundle is a statistical manifold in the sense of Riemannian.
Keywords: statistical structure, symmetric connection, Finsler metric, lift, tangent bundle.

Introduction

The notion of statistical structure for Riemannian spaces are studied in detail in [A-N00], [M98], [M-I03] and [M-G93]. In [A-N00] and [M-I03] the geometry of statistical manifolds are studied, for example, dual connections, information geometry and almost complex statistical structure and so on. In particular, the complete-, horizontal-, vertical-lifts of metrics and connections of statistical manifolds to its tangent bundle play an important role in the study of statistical structures for the base manifolds and its tangent bundles, for examples, Theorem 1.1 and 1.2 in §1. Further, Prof.Hasegawa and Prof.Yamauchi obtain the conditions to be of constant curvature for the given statistical manifold by studying its statistical tangent bundle with the complete lift of the metric on the base manifold in [H-Y05], recently.

One can see the usefulness of studying lifted geometrical structures on its tangent bundle. So the author studies the conditions that its tangent bundle, which has lifted vertical-, horizontal-, or complete-lift metric, is a statistical manifold in the sense of Riemann for the given Finsler spaces, and obtains some conditions for the Finsler spaces. The obtained conditions, however, are negative for Finsler geometry. The author thinks that the condition "*symmetric*" in the sense of Riemannian for the connection is very rigid, in the future, if we study the notion of statistical structure for Finsler spaces, this symmetric condition must be substituted by another condition.

In §1, general preparation and the notion of the statistical structure are stated. In §2, the notion of Finsler spaces and Finsler connections are stated and the relation of between Finsler torsion tensor fields and Riemannian torsion tensor fields of the given Finsler connection are pointed out. In §3, in §4 and in §5 the vertical-, the horizontal- and the complete-lifts of the Finsler metric are given and the conditions that its tangent bundle comes to a statistical manifold are shown, respectively.

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In present paper, the author obeys [M86] with respect to the notation of the covariant derivation and the position of its indices and refers [Y-I73] with respect to the various lifts to its tangent bundles.

1 Statistical Structure of Riemannian Cases

Let M be an $n-$dimensional differentiable manifold, h a semi-Riemannian metric on M, and ∇ a symmetric affine connection on M. A *statistical structure* on M is a pair (h, ∇) such that ∇h is totally symmetric, and (M, h, ∇) is called a *statistical manifold.*

On a statistical manifold (M, h, ∇), the symmetric tensor field $K := \nabla h$ is called the *cubic form* or *skewness field* and written in the local coordinate system (x^i) of M as follows:

$$K = (K_{ijk}) = K_{ijk} dx^i \otimes dx^j \otimes dx^k, \tag{1.1}$$

where $K_{ijk} := h_{ij,k}$ and $,k$ is covariant derivative of $\frac{\partial}{\partial x^k}$.

Further we define the tensor field S of type (1,2) by

$$h(S(X,Y),Z) := K(X,Y,Z) = (\nabla_Z h)(X,Y) \tag{1.2}$$

for any $X, Y, Z \in \mathcal{X}(M)$. The tensor field S is called the *skewness operator* of (M, h, ∇) and written as follows:

$$S = (S^k_{ij}) = S^k_{ij} \frac{\partial}{\partial x^k} \otimes dx^i \otimes dx^j, \tag{1.3}$$

where $S^k_{ij} := K_{aij} h^{ak}$ and $(h^{ij}) := (h_{ij})^{-1}$.

Next, let ∇^0 be the Levi-Civita connection of h. Then the following relation

$$\nabla - \nabla^0 = -\frac{1}{2} S \tag{1.4}$$

is satisfied.

In general, we can define the *dual connection* ∇^* of ∇ as follows:

$$Zh(X,Y) = h(\nabla_Z X, Y) + h(X, \nabla^*_Z Y). \tag{1.5}$$

Then $\nabla^* h$ is totally symmetric, namely (h, ∇^*) is another statistical structure of M, and the relation

$$\nabla^0 = \frac{1}{2}(\nabla + \nabla^*) \tag{1.6}$$

is also satisfied.

Let (x^i, y^i) be the induced coordinate system of TM. Then various lifts of h are defined as follows:

$$\begin{aligned} h^c &= y^k \partial_k h_{ij} dx^i \otimes dx^j + h_{ij}(dy^i \otimes dx^i + dx^i \otimes dy^j) \quad \text{complete lift} \\ &= y^k h_{ij,k} dx^i \otimes dx^j + h_{ij}(\delta y^i \otimes dx^i + dx^i \otimes \delta y^j), \quad (\delta y^i = dy^i + y^a \Gamma^i_{ar} dx^r) \\ h^h &= (y^a \Gamma^k_{ai} h_{jk} + y^a \Gamma^k_{aj} h_{ki}) dx^i \otimes dx^j + h_{ij}(dy^i \otimes dx^i + dx^i \otimes dy^j) \quad \text{horizontal lift} \\ &= h_{ij}(\delta y^i \otimes dx^i + dx^i \otimes \delta y^j) \\ h^v &= h_{ij} dx^i \otimes dx^j \quad \text{vertical lift} \end{aligned} \tag{1.7}$$

and the various lifts of a vector field $X = X^i\partial_i$ on M are defined as follows:

$$
\begin{aligned}
X^c &= X^i\partial_i + y^a(\partial_a X^i)\partial_{\bar{i}} \quad \text{complete lift} \\
&= X^i\delta_i + y^a X^i_{,a}\partial_{\bar{i}}, \quad (\delta_i = \partial_i - y^a\Gamma^k_{ai}\partial_{\bar{k}}) \\
X^h &= X^i\partial_i - y^a\Gamma^k_{ai}X^i\partial_{\bar{k}} \quad \text{horizontal lift} \\
&= X^i\delta_i \\
X^v &= X^i\partial_{\bar{i}} \quad \text{vertical lift}
\end{aligned} \tag{1.8}
$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^k_{ij} are the coefficients of ∇.

Further the complete lift ∇^c of ∇ is defined as follows:

$$\nabla^c_{X^c}Y^c = (\nabla_X Y)^c \tag{1.9}$$

and the horizontal lift ∇^h of ∇ is defined as follows:

$$\nabla^h_{X^h}Y^h = (\nabla_X Y)^h,\ \nabla^h_{X^h}Y^v = (\nabla_X Y)^v,\ \nabla^h_{X^v}Y^h = 0,\ \nabla^h_{X^v}Y^v = 0 \tag{1.10}$$

for any $X,\ Y \in \mathcal{X}(M)$.

Then the following theorems are established.

Theorem 1.1 ([M-G93], [M-I03]) *Let (M, h, ∇) be a statistical manifold. Then (TM, h^c, ∇^c) is a statistical manifold with neutral metric. The cubic form of (TM, h^c, ∇^c) is the complete lift K^c of K, where*

$$
\begin{aligned}
K^c =& y^a\partial_a K_{ijk}dx^i \otimes dx^j \otimes dx^k \\
&+ K_{ijk}(dy^i \otimes dx^j \otimes dx^k + dx^i \otimes dy^j \otimes dx^k + dx^i \otimes dx^j \otimes dy^k).
\end{aligned} \tag{1.11}
$$

Theorem 1.2 ([M-I03]) *Let (M, h, ∇) be a statistical manifold. Then (TM, h^h, ∇^h) is a statistical manifold if and only if $\nabla h = 0$.*

2 Finsler Spaces and Finsler Connections

Let F be a positively differentiable function on $T_0(M) = TM\backslash\{0\}$. We assume that F is (1)p-homogeneous with respect to y^i and the matrix $(g_{ij}) = (\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j})$ is regular. Then (M, F) is called a *Finsler space.*

Let $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ be a Finsler connection of (M, F). Hereafter we take the adapted frames $(\delta_i, \partial_{\bar{i}}) = (\partial_i - N^r_i\partial_{\bar{r}}, \partial_{\bar{i}})$ as the frames of $T_0(M)$. Then the Finsler connection $\nabla = (N^i_j, F^i_{jk}, C^i_{jk})$ satisfies the following formulae:

$$\nabla_{\delta_j}\delta_i = F^r_{ij}\delta_r, \quad \nabla_{\delta_j}\partial_{\bar{i}} = F^r_{ij}\partial_{\bar{r}}, \quad \nabla_{\partial_{\bar{j}}}\delta_i = C^r_{ij}\delta_r, \quad \nabla_{\partial_{\bar{j}}}\partial_{\bar{i}} = C^r_{ij}\partial_{\bar{r}} \tag{2.1}$$

and ∇ is regarded as an affine connection of $T_0(M)$. Further for the coframes $(dx^i, \delta y^i) = (dx^i, dy^i + N^i_j dx^j)$,

$$\nabla_{\delta_j}dx^i = -F^i_{rj}dx^r,\ \nabla_{\delta_j}\delta y^i = -F^i_{rj}\delta y^r,\ \nabla_{\partial_{\bar{j}}}dx^i = -C^i_{rj}dx^r,\ \nabla_{\partial_{\bar{j}}}\delta y^i = -C^i_{rj}\delta y^r \tag{2.2}$$

are satisfied.

Here we assume that the Finsler connection ∇ is symmetric, namely, torsion free as an affine connection of $T_0(M)$. We write the coefficients of torsion tensor field $\bar{T}$ of ∇ with respect to the adapted frames $\partial_A = (\delta_i, \partial_{\bar{i}})$ by $\bar{T}^A_{BC}$, where the indices $i, j, k = 1, 2, \cdots, n; A, B, C = 1, 2, \cdots, n, n+1, \cdots, 2n : \bar{1} = n+1, \bar{2} = n+2, \cdots, \bar{n} = 2n$. Since the following formulae of the torsion tensor field of ∇

$$\bar{T}(\partial_B, \partial_C) = \nabla_{\partial_B}\partial_C - \nabla_{\partial_C}\partial_B - [\partial_B, \partial_C] = \bar{T}^A_{BC}\partial_A, \tag{2.3}$$

we have

$$\begin{aligned} &\bar{T}^i_{jk} = F^i_{kj} - F^i_{jk},\ \bar{T}^{\bar{i}}_{jk} = -\delta_k N^i_j + \delta_j N^i_k,\ \bar{T}^i_{j\bar{k}} = -C^i_{jk},\ \bar{T}^{\bar{i}}_{j\bar{k}} = F^i_{kj} - \partial_{\bar{k}} N^i_j \\ &\bar{T}^i_{\bar{j}k} = C^i_{kj},\ \bar{T}^{\bar{i}}_{\bar{j}k} = \partial_{\bar{j}} N^i_k - F^i_{jk},\ \bar{T}^i_{\bar{j}\bar{k}} = 0,\ \bar{T}^{\bar{i}}_{\bar{j}\bar{k}} = -C^i_{jk} + C^i_{kj}. \end{aligned} \tag{2.4}$$

According to our assumption ($\bar{T} = 0$), all torsion tensor fields T, R, P, S, C of Finsler connection ∇ of (M, F) vanish as follows:

$$T^i_{jk} = 0,\ R^i_{jk} = 0,\ P^i_{jk} = 0,\ S^i_{jk} = 0,\ C^i_{jk} = 0, \tag{2.5}$$

where $T^i_{jk} = F^i_{jk} - F^i_{kj},\ R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k,\ P^i_{jk} = \partial_{\bar{k}} N^i_j - F^i_{kj},\ S^i_{jk} = C^i_{jk} - C^i_{kj}$.

3 Vertical lifts of Finsler metrics

Let (M, F, ∇) be a Finsler space with a Finsler connection ∇, which satisfies (2.5),namely, to be torsion free as an affine connection ∇ of $T_0(M)$.

Now we consider the vertical lift of the Finsler fundamental tensor field $g_{ij}(x, y)$ as follows:

$$g^v = g_{ij} dx^i \otimes dx^j. \tag{3.1}$$

Then for any vector field $Z = Z^i\delta_i + Z^{\bar{i}}\partial_{\bar{i}}$ of $T_0(M)$, we have

$$\begin{aligned} \nabla_Z g^v &= Z^k \nabla_{\delta_k} g^v + Z^{\bar{k}} \nabla_{\partial_{\bar{k}}} g^v \\ &= (Z^k g_{ij|k} + Z^{\bar{k}} g_{ij}|_k) dx^i \otimes dx^j, \end{aligned} \tag{3.2}$$

where $g_{ij|k} = \delta_k g_{ij} - F^r_{ik} g_{rj} - F^r_{jk} g_{ir},\ g_{ij}|_k = \partial_{\bar{k}} g_{ij} - C^r_{ik} g_{rj} - C^r_{jk} g_{ir}$.

Therefore for any vector fields $X = X^i\delta_i + X^{\bar{i}}\partial_{\bar{i}},\ Y = Y^i\delta_i + Y^{\bar{i}}\partial_{\bar{i}}$

$$\begin{aligned} K(X, Y, Z) = \nabla_Z g^v(X, Y) &= (Z^k g_{ij|k} + Z^{\bar{k}} g_{ij}|_k) X^i Y^j \\ &= g_{ij|k} X^i Y^j Z^k + g_{ij}|_k X^i Y^j Z^{\bar{k}} \end{aligned} \tag{3.3}$$

is satisfied.

Here we assume that $K = \nabla g^v$ is *totally symmetric*. From (3.3) and the arbitrariness of X, Y, Z

$$g_{ij|k}\ :\ \text{totally symmetric} \quad \text{and} \quad g_{ij}|_k = 0 \tag{3.4}$$

are satisfied. Thus we have

Theorem 3.1 *Let* (M, F, ∇) *be a Finsler space with a Finsler connection* ∇ *satisfied* (2.5). *Then* $(T_0(M), g^v, \nabla)$ *is a statistical manifold if and only if* (3.4) *are satisfied.*

From $C^i_{jk} = 0$ of (2.5), however, we have $g_{ij}\big|_k = \partial_{\bar{k}} g_{ij}$ so that $g_{ij}\big|_k = 0$ implies $g_{ij}(x, y) = g_{ij}(x)$. Thus we have

Theorem 3.2 *Let* (M, F, ∇) *be a Finsler space with a Finsler connection* ∇ *satisfied* (2.5). *If* $(T_0(M), g^v, \nabla)$ *is a statistical manifold, namely,* ∇g^v *is totally symmetric, then* g *is a Riemannian metric.*

4 Horizontal lifts of Finsler metrics

Next we consider the horizontal lift of the Finsler fundamental tensor field $g_{ij}(x, y)$ as follows:

(4.1)
$$\begin{aligned} g^h &= (N^k_i g_{jk} + N^k_j g_{ki}) dx^i \otimes dx^j + g_{ij}(dy^i \otimes dx^i + dx^i \otimes dy^j) \\ &= g_{ij}(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j) \end{aligned}$$

Then for any vector field $Z = Z^i \delta_i + Z^{\bar{i}} \partial_{\bar{i}}$ of $T_0(M)$, we have

(4.2)
$$\begin{aligned} \nabla_Z g^h &= Z^k \nabla_{\delta_k} g^h + Z^{\bar{k}} \nabla_{\partial_{\bar{k}}} g^h \\ &= Z^k g_{ij|k}(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j) + Z^{\bar{k}} g_{ij}\big|_k (\delta y^i \otimes dx^j + dx^i \otimes \delta y^j) \\ &= (Z^k g_{ij|k} + Z^{\bar{k}} g_{ij}\big|_k)(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j) \end{aligned}$$

Therefore for any vector fields $X = X^i \delta_i + X^{\bar{i}} \partial_{\bar{i}}$, $Y = Y^i \delta_i + Y^{\bar{i}} \partial_{\bar{i}}$

(4.3)
$$K(X, Y, Z) = \nabla_Z g^h(X, Y) = (Z^k g_{ij|k} + Z^{\bar{k}} g_{ij}\big|_k)(X^{\bar{i}} Y^j + X^i Y^{\bar{j}})$$

is satisfied.

Here we assume that $K = \nabla g^h$ is *totally symmetric.* From (4.3) and the arbitrariness of X, Y, Z

(4.4)
$$g_{ij|k} = 0 \quad \text{and} \quad g_{ij}\big|_k = 0$$

are satisfied. Thus we have

Theorem 4.1 *Let* (M, F, ∇) *be a Finsler space with a Finsler connection* ∇ *satisfied* (2.5). *Then* $(T_0(M), g^h, \nabla)$ *is a statistical manifold if and only if* (4.4) *are satisfied.*

However, the relation $g_{ij}\big|_k = 0$ implies $g_{ij}(x, y) = g_{ij}(x)$ by the same reason in §3. Thus we have

Theorem 4.2 *Let* (M, F, ∇) *be a Finsler space with a Finsler connection* ∇ *satisfied* (2.5). *If* $(T_0(M), g^h, \nabla)$ *is a statistical manifold, namely,* ∇g^h *is totally symmetric, then* g *is a Riemannian metric.*

5 Complete lifts of Finsler metrics

Lastly, we consider the complete lift of the Finsler fundamental tensor field $g_{ij}(x,y)$ as follows:

$$g^c = y^k g_{ij|k} dx^i \otimes dx^j + g_{ij}(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j). \tag{5.1}$$

Then for any vector field $Z = Z^i\delta_i + Z^{\bar{i}}\partial_{\bar{i}}$ of $T_0(M)$, we have

$$\begin{aligned}
\nabla_Z g^c &= Z^k \nabla_{\delta_k} g^c + Z^{\bar{k}} \nabla_{\partial_{\bar{k}}} g^c \\
&= Z^k (y^r g_{ij|r})_{|k} dx^i \otimes dx^j + Z^k g_{ij|k}(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j) + \\
&+ Z^{\bar{k}} (y^r g_{ij|r})\big|_k dx^i \otimes dx^j + Z^{\bar{k}} g_{ij}\big|_k (\delta y^i \otimes dx^j + dx^i \otimes \delta y^j) \\
&= (Z^k (D^r_k g_{ij|r} + g_{ij|0|k}) + Z^{\bar{k}} (g_{ij|k} + C^r_{0k} g_{ij|r} + g_{ij|0}\big|_k)) dx^i \otimes dx^j + \\
&+ (Z^k g_{ij|k} + Z^{\bar{k}} g_{ij}\big|_k)(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j)
\end{aligned} \tag{5.2}$$

Therefore for any vector fields $X = X^i\delta_i + X^{\bar{i}}\partial_{\bar{i}}$, $Y = Y^i\delta_i + Y^{\bar{i}}\partial_{\bar{i}}$

$$\begin{aligned}
K(X,Y,Z) &= \nabla_Z g^c(X,Y) \\
&= (Z^k (g_{ij|r} D^r_k + g_{ij|0|k}) + Z^{\bar{k}} (g_{ij|k} + g_{ij|r} C^r_{0k} + g_{ij|0}\big|_k)) X^i Y^j + \\
&+ (Z^k g_{ij|k} + Z^{\bar{k}} g_{ij}\big|_k)(X^{\bar{i}} Y^j + X^i Y^{\bar{j}}) \\
&= (g_{ij|r} D^r_k + g_{ij|0|k}) X^i Y^j Z^k + g_{ij|k}(X^i Y^j Z^{\bar{k}} + X^i Y^{\bar{j}} Z^k + X^{\bar{i}} Y^j Z^k) + \\
&+ (g_{ij|r} C^r_{0k} + g_{ij|0}\big|_k) X^i Y^j Z^{\bar{k}} + g_{ij}\big|_k (X^{\bar{i}} Y^j + X^i Y^{\bar{j}}) Z^{\bar{k}}
\end{aligned} \tag{5.3}$$

is satisfied, where $D^i_j = y^r F^i_{rj} - N^i_j$. We put

$$E_{ijk} = g_{ij|r} D^r_k + g_{ij|0|k}, \quad W_{ijk} = g_{ij|r} C^r_{0k} + g_{ij|0}\big|_k. \tag{5.4}$$

Here we assume that $K = \nabla g^c$ is *totally symmetric*. From (5.3) and the arbitrariness of X, Y, Z

$$\begin{aligned}
&E_{ijk} \text{ and } g_{ij|k} \text{ : totally symmetric} \\
&W_{ijk} = 0 \text{ and } g_{ij}\big|_k = 0
\end{aligned} \tag{5.5}$$

are satisfied. Thus we have

Theorem 5.1 *Let (M, F, ∇) be a Finsler space with a Finsler connection ∇ satisfied* (2.5)*. Then $(T_0(M), g^c, \nabla)$ is a statistical manifold if and only if* (5.5) *are satisfied.*

However, the relation $g_{ij}\big|_k = 0$ implies $g_{ij}(x,y) = g_{ij}(x)$ by the same reason in §3. Thus we have

Theorem 5.2 *Let (M, F, ∇) be a Finsler space with a Finsler connection ∇ satisfied* (2.5)*. If $(T_0(M), g^c, \nabla)$ is a statistical manifold, namely, ∇g^c is totally symmetric, then g is a Riemannian metric.*

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