Applications of Leontief's Input-Output Analysis in Our Economy

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"Input-output analysis is a basic methods of quantitative economics that portrays macroeconomic activity as a system of interrelated goods and services. The analysis usually involves constructing a table in which each horizontal row describes how one industry's total product is divided among various productive processes and final consumption. Each vertical column denotes the combination of productive resources used within one industry. Each figure in any horizontal row is also figure in a vertical column. Input-0utput tables can be constructed for whole economies or for segments within economies."

1. INTRODUCTION

Many words in economics have been used in different senses by different writers, at different times, and in different contexts. The word input-output economics is a case in point. Input-output economics deals with aggregate categories. It falls within the purview of macroeconomics which studies the behavior of the aggregate economy. In macroeconomics the unit of analysis is the national economy. The term interindustry analysis is also used, since the fundamental purpose of the input-output framework is to analyze the interdependence of industries or sectors in an economy. Yet because it is applied within the realm of observable and measured phenomena, it is also considered a branch of econometrics which was built on macroeconomic theory and became part of mainstream economics. If we interprete the meaning of econometrics literally as such that it would be economic measurement or perhaps measurement in economics, input-output analysis is clearly a case of economic measurement and therefore solidly in the mainstream of econometrics which nowadays heavily depends on input-output technique in econometric model building and operation. An economist used input-output data as the basis for testing empirically a variety of assertions made about certain tendency cies in the economic history of industrializing economies.

As such, much of the existing literature on the subject is highly technical in nature. That is why input-output analysis is called quantitative economics.

In his input-output economics Leontief sets out to pursue a dictum that has served as a guiding thread throughout his career: that economic concepts were of little validity unless they could be observed and measured and he was convinced that not only was well-formulated theory of utmost importance but so too was its application to real economics. Leontief himself says in his book on Input-Output Economics as follows;

As a result we have in economics today a high concentration of theory without fact on the one hand, and a mounting accumulation of fact without theory on the other. The task of filling the "empty boxes of economic theory" with relevant empirical content becomes every day more urgent and challenging.

Leontief demonstrated how to combine economic facts and theory known as interindustry or input-output analysis. This is what underlying Leontief's input-output economics is all about. Therefore Leontief's input-output analysis does not come with great deal of theoretical baggage that is hard to prove in real life. Of course, it is supceptible to distortions from measurement error or inaccurate modeling, but its underlying strength lies in being driven by real data. In such a sense, input-output analysis remains an active branch of economics. Input-output economics provides us with a powerful economic analysis tool in the form of input-output analysis.

Input-output analysis can be regarded as a vast collection of data describing out economic system, and/or as an analytical technique for explaining and predicting the behavior of our economic system. The *sine qua non* of empirical input-output work is the input-output table, reminiscent of Quesnay's *tableau economique*, which represents the circulation of commodities in an economic system and can be considered the first input-output schema ever to have been formulated. Quesnay was one of the leading theoreticians whose work inspired the formation of a group of French agrarian social reformers called the physiocrats. Though its central focus was on agriculture, the tableau was a diagrammatic representation of how expenditures can be traced through an economy in a systematic way and also represented a basic working model of an economy and its extended reproduction. It highlighted the processes of production, circulation of money and commodities, and the distribution of income. And, despite having originally appeared in cumbersome zigzag form, the Tableau's easy adaptability to Leontief's input-output or double-entry table format has been demonstrated. 1/

There are two applications of the Leontief model: an open model and a closed model. An open model finds the amount of production needed to satisfy an increase in demand whereas the closed model deals only with the income of each industry. The closed model means that all inputs into production are produced and all outputs exist merely to serve as input. That is to say that all outputs are also used as inputs. Industries produce commodities using commodities as well as factor inputs. Housholds produce these factor inputs using commodities. And as a matter of fact, the Leontief Open Production Model provides us with a powerful economic analysis tool in the form of input-output analysis. Nowadays, Many people apply the input-output methodology to empirical problems requiring economic analysis. The real strength of the input-output methodology lay in its practical uses as an implement of economic analysis.

The input-output accounts which are composed of the data sets used in input-output analysis were included as an integral part of the SNA to represent the structural characteristics of the economy. The SNA is a system of accounts, one component of which is the input-output accounts. An SNA is a square matrix where the number of rows or columns equals the number of accounts and organizes various accounts into one table, making use of the fact every transaction is both a receipt to one party and an expenditure to another. A principle of the SNA is that every account comprises a row and a column. The row lists the receipts and a column the expenditures. The first account consists of the first row and the first column. Thus accounts are not written separately as pairs of columns but collapsed into each other by means of a matrix. The advantage of this organization is that it admits a bird's eye view of an entire system of national accounts. National accounts are organized in order to measure the national product or income of an economy. Product and income are different concepts and it takes a framework to relate the two through the economic activities of production, consumption, and distribution.

The first SNA was initiated and published by the United Nations in 1953, followed by revision, the third being published in 1968. The SNA is a way to portray clealy and concisely a framework within which the statistical information needed to analyze the economic process in all its many aspects could be organized and related. The inclusion of input-output accounts as part of the SNA contributed to the spread of input-output work throughout the world. The power of input-output analysis is its capacity to analyze economies as they are given by a coherent set of data, namely the national accounts.

This article is concerned with examining the basic structure of input-output model as a tool of economic analysis and exploring how to develop Leontief's input-output model from the basic transaction table. We also demonstrate our effort to combine economic facts and economic theory known as input-output analysis using a simple numerical example. Gaining this basic understanding is the purpose of the simple example.

We begin to investigate thr fundamental structure of the input-output model, the assumption behind it, and also the simplest kinds of problems to which it is applied. Doing excercises attests usefulness of the input-output technique as an indispensable tool of economic analysis.

2. INPUT-OUTPUT ANALYSIS

Input-output analysis is the name given to an analytical framework developed by Wassily Leontief. One often speaks of a Leontief model when referring to input-output. The term interindustry analysis is also used, since the fundamental purpose of the input-output framework is to analyze the interdependence of industries in an economy.

Input-output analysis, or the quantitative analysis of interindustry relations, is a way of describing the allocation of resources in a multisectoral economy. The data of input-output analysis are the flows of goods and services inside the economy that underlie the summary statistics by which economic activity is conventionally measured. The great virtue of input-output analysis is that it surfaces the indirect internal transactions of an economic system and brings them into the reckonings of economic theory.

Like most successful innovations, input-output analysis has developed from a basic idea of great simplicity: all transactions that involve the sale of products or services within an economy during a given period are arrayed in a square indicating simultaneously the sectors making and the sectors receiving delivery. More specifically, every row in an input-output table shows the sales made by one economic sector to every other sector, and every column shows what each economic sector purchased from every other sector. The nature of the table and the individual entries are obviously determined by the number and definition of the sectors distinguished. Most of the current input-output tables divide the commodity-producing sector (including transportation and service) very finely-in the most elaborate tables into more than 400 industries-so that interindustry relations, i. e. sales of interindustry products between industries, can be followed in great detail

Input-output analysis is a very useful framework for examining changes in the structure of an economy over time, particularly if a series of comparable tables are available for the economy of interest. Input-output analysis focuses attention on the flows of outputs and inputs among the various sectors of the system. It is frequently used as an aid in regional or national economic planning, because it is capable of revealing the impacts of decisions or shocks in all sectors, fully accounting for their inter-related and balanced nature.

In fact, the sectors of an economy are linked together. The production of many final goods requires not only the primary factors of labor and capital, but the outputs of other sectors as intermediate goods. For instance, the manufacture of automobiles requires the intermediate goods of tires and headlights, which, in turn, require the intermediate goods of rubber and glass, respectively. Thereore, the total demand for any product, (e.g. tires), will be equal to the sum of all the intermediate demands (e.g. by automobile manufactures) and final demand (e.g. by consumers and firms purchasing tires directly). Input-output models account for the linkages across the secors or industries of our economy.

3. NATIONAL INCOME ACCOUNTS AND INPUT-OUTPUT TABLES

In the input-output table, the overall economic activities of the economy are systematically summarized. National-income accounting is also a systematic summary of economic activites, although it differs from input-output in detail and method. The national-income accounts register all the final goods and services produced in the economy during a certain year by the four sectors: business, personal, government, and "rest of the world."

The corresponding data are found in the “value-added” row of an input-output table. The input-output table, however, often divides business sector into many industries. It consists of intermediate product flows bordered to the right by one or more vectors of final demand and below by one or more vectors of primary inputs and other production costs, such as provisions for depreciation and indirect taxes.

Gross Domestic Product (GDP) is also recorded by the nature of expenditures. It is divided into personal consumption, capital formation, government expenditure and net exports in a table of the national-income accounts. In an actual input-output table, final goods and services are often divided into the same categories as those in the national-income account. Four columns are used to describe how the final goods and services produced by each sector are allocated to these four different uses.

In the national-income accounts, Net Domestic Product (NDP) and National Income (NI) are also defined.

NDP= GDP － depreciation

Depreciation is the loss in value of capital equipment resulting from wear and tear and obsolescence.

NI= NDP － indirect business taxes

When data are available, the value-added row in an input-out table can always be decomposed into wage, depreciation, indirect business taxes and profit or loss. Thus practically all the information contained in the national-income accounts is also obtainable from an input-output table.

However, the converse is not true. The national-income accounts are concerned with national totals, not with individual industries. Thus detailed information on transactions among industries is not available. As a result, when there are changes in individual industries constituting an economy, they are not detectable in the national-income accounts. This shortcoming is eliminated when the input-output table is used; its breakdown of the economy is in great detail, and changes in the individual components of an aggregate variable are systematically recorded.

Input-output tables present a comprehensive portrayal of sales and purchases by each industry in the economy. Because transactions are arranged in matrix form, each cell represents simultaneously a sale and a purchase. Along each row, the sale by an industry to each intermediate and final user is shown. Final users include private consumers, public consumers (government), private and public investors, and foreign traders. It is the total of these sales to final users that represents the gross domestic product (GDP). Total output in the economy is composed of the GDP plus all sales to intermediate users (such as agriculture, mining, manufacturing, and services).

The national income serves as the empirical basis for macroanalysis which seeks to see "the forest and not the trees." However, changes in individual "trees" sometimes bring about substantial change in "the forest." As a whole, in an input-output table, the value-added row and the final goods-and-services columns enable us to see both the forest and the trees.

4. INPUT-OUTPUT TECHNIQUE

The analysis usually involves constructing a table in which each horizontal row describes how one industry's total product is divided among various production processes and final demand. Each vertical column denotes the combination of ptoductive resources used within one industry. In each column of input-output table, purchases from intermediate producers and primary factors of production (labor, capital, and land) are recoreded.

Input-output table has one row and one column for each sector of the economy and shows, for each pair of sectors, the amount or value of goods and services that flowed directly between them in each direction during a stated period. Typically, the tables are arranged so that the entry in the rth row and cth column gives the flow from the rth sector to the cth sector (here r and c refer to any two numbers, such as 1, 2, etc.)

If the sectors are defined in such a way that the output of each is fairly homogeneous, they will be numerous. The amount of effort required to estimate the output of each sector, and to distribute it among the sectors that uses it, is prodigious. This phase of input-output work corresponds in its general descriptive nature to the national income accounts. The complete specification of all interindustry transactions distinguishes input-output acconunts from national income and product accounts and helps to bridge the macro and sectoral components of an economy. The double-counting in input-output accounts provides detailed information for analysis and planning purposes.

Many important accounting balances must be maintained in constructing an input-output account. The first major accounting balance is that total outlays by an industry (the total of elements in a column) must equal total output of the industry-total sales of output of the industry to all intermediate and final users (the total of elements in the row for the respective industry). Differences between these two totals helps input-output accountants identify problems with the basic data collected by surveys, censuses, and other means.

The second major accounting balance is that the sum of all income earned by the factors of production (gross income received) must be equal to the sum of all expenditures made by final users (gross domestic product). This accounting balance ensures that all income recorded as received is also showm as being spent.

The analytical phase of input-output work has been built on a foundation of two piers. The first pier is a set of accounting equations, one for each industry. The first of these equetions says that the total output of the first industry is equal to the sum of the separate amounts sold by the first industry to the other industries; the second equation says the same thing for the the second industry; and so on. Thus the equation for any industry says that its total output is equal to the sum of all the entries in that industry's row in the input-output table.

The second pier is another set of equations, at least one for each industry. The first group of these equations shows the relationships between the output of the first industry and the inputs it must get from other industries in order to produce its own output; the others do the same for the second and all other industries.

Work in input-output economics may be purely desctiptive, dealing only with the preparation of input-output tables. Or it may be purely theoretical, dealing with the formal relationships that can be derived under various assumptions from the equations just mentioned. Or it may be a mixture, using both empirical data and thoretical relationships in the attempt to explain or predict actual developments.

5. THE INPUT-OUTPUT TABLE

The input-output (I-O) table describes intersectoral flows in a tabular form and records the purchases and sales across the sectors of an economy over a given period of time. Suppose an economic system or region has a total of n production sectors. The output of a given sector is used by intermediate demanders (the production sectors use each other's output in their production activities) and by final demanders (typically households, the government, and other regions or nations that trade with the given system). We present a transaction table of such an economy in Figure 1 that represents a basic Input-Output Model with non-competitive Imports. Non-competitive imports include products that are either not producible or not yet produced in the country. The value of goods imported is recorded as a separate row in the transactiona matrix. 2/ Therefore the example in Figure 1 has no corresponding column since no equivalent products are produced domestically.

In this table, X_i is the gross output of the ith sector, X_{ij} represents the amount of the ith sector's output used by the jth sector to produce its output, and X_j is the final demanders' use of the ith sector's output. The use of primary inputs such as labor, W, and capital, R is described in the bottom rows of the table. In those rows W_i, represents the use of labor in the production of ith product, W is the use of labor by final demanders, R_i is the use of capital in the production of other goods, and R is the final demand for capital.

The rows of the table describe the deliberies of the total amount of a product or primary input to all uses, both intermediate and final. For example, suppose sector 1 represents food products. Then the first row tells us that, out of a gross output of X_1 tons of food products, an amount X_{11} is used in the production of food products themselves, an amount of X_{12} must be delivered to sector 2, X_{1i} tons are delivered to sector i, X_{1n} to sector n, and X_1 tons are consumed by final end users of food products.

The columns of the table describe the input requirements to produce the gross output totals. Thus, producing the X_1 tons of food products requires X_{11} tons of food products, along with X_{21} units of output from sector 2 (steel, perhaps), X_{i1} from sector i, X_{n1} from sector n, W_1 hours of labor, and R_1 dollars of capital. An entry of 0 in one of the cells of the table indicates that none of the product represented by the row is required by the product represented by the column, so none is delivered.

Below we set out a basic input-output (I-O) table under the following key assumptions:

1. Each sector or industry is characterized by a fixed coefficients production function. That is, there is a fixed or inflexible relationship between the level of output of any sector and the levels of required inputs. Irespective possible. Economies of scale in production are thus ignored. For example, as we all know that the degree to which a firm can substitute the factors of production is reflected in the shape of isoquants. Using the two factor Cobb-Douglas production function geometry, we can see the isoquant curves of constant output and the right-angle isoquant (the isoquants are "square") repre-

sents a fixed-coefficients production function which shows that elasticity of substitution is zero.

2. The production of output in each sector is characterized by constant returns to scale. That is, an r% increase (decrease) in the output of a sector requires an r% increase (decrease) in all of the inputs. Production in a Leontief system operates under what is known as constant returns to scale. Returning to the two factor Cobb-Douglas production function, we can see the sum of transformation parameters, the exponents a and b, indicates the returns to scale. That is, r = a+b. To demonstrate, starting with a general Cobb = Douglas Production function, $Q = AK^aL^b$, multiplying the inputs of capital and labor by a constant c gives:
$A(cK)^a(cL)^b = Ac^{a+b}K^aL^b = c^{a+b}Q$
If a+b = 1, then we have constant returns to scale.

3. Technology is given. The fixed coefficients production functions are set and reflect a given state of technology.

4. Each industry produces only one homogeneous commodity.
(Broadly interpreted, this does not permit the case of two or more jointly produced commodities, provided they are produced in a fixed proportion to one another.)

In constructing an input-output (I-O) table, the entries can be in physical units (e.g., tons of steel or hours of service) or in terms of monetary value (e.g., dollars or yen). We will use monetary value and assume constant unit prices for inputs and outputs in order to have fixed relationships between monetary values and physical quantities. Doing so greatly facilitates the interpretation of the I-O table and the derivation of the I-O relationships.

The I-O table can be divided vertically according to the type of demand (interindustry demands and final demands) and horizontally according to the type of input (domestic intermediate goods, domestic primary factors of production, and imports). The rows of such a table describe the distribution of a producer's output throughout the economy. The columns describe the composition of inputs required by a particular industry to produce its output. These interindustry exchanges of goods constitute the endogenous division of the Table. The additional columns, labeled Final Demand, record the sales by each sector to final markets for their production. The additional rows, labeled Value Added, account for the other (nonindustrial) inputs to production.

In this general model presented here we will consider n sectors or industries, two primary factors of production (capital and labor), and initially four types of final demand (personal consumption expenditures, C; investment expenditures, I; government purchases of goods and services, G; and exports, E).

Referring to Figure 1, the Xs indicate the value of output.
For example,

$$X_i = \text{value of the output of sector } i \ (i = 1 \cdots n)$$

Figure 1. General Inut-Output Table

Purchases by:		Intermediate Users Sectors/Industries					Final Demands				Total Demand
		1	2	3	···	n	C	I	G	E	X
Sales by:	1	X_{11}	X_{12}	X_{13}	···	X_{1n}	C_1	I_1	G_1	E_1	X_1
	2	X_{21}	X_{22}	X_{23}	···	X_{2n}	C_2	I_2	G_2	E_2	X_2
	3	X_{31}	X_{32}	X_{33}	···	X_{3n}	C_3	I_3	G_3	E_3	X_3
Sectors/											
Industries	•	•	•	•	···	•	•	•	•	•	?
	•	•	•	•	···	•	•	•	•	•	?
	n	X_{n1}	X_{n2}	X_{n3}		X_{nn}	C_n	I_n	G_n	E_n	X_n
Value-	W	W_1	W_2	W_3	···	W_n	W_C		W_G		W
Added	R	R_1	R_2	R_3	···	R_n					R
Imports	M	M_1	M_2	M_3	···	M_n	M_C	M_I	M_G		M
Total Supply	X	X_1	X_2	X_3	···	X_n	C	I	G	E	

where

X_i = value of the output of sector i (i = 1・・・n)

X_{ij} = sales by sector i to sector j, or the value of inputs from sector i used to produce the output of sector j (i = 1・・・n; j = 1・・・n). It represents the amount of the ith sector' output used by the jth sector to produce its output.

W_j = wages in sector j (j = 1・・・n). It represents the use of labor in the production of the ith product.

R_j = interest and profits in sector j

M_j = imports of sectors j

C_j = personal consumption expenditures for the output of sector i

I_j = investment expenditures for the output of sector i

G_j = government purchases of the output of sector i

E_j = exports of the output of sector i

M_C, M_I and M_G = imports of final goods by consumers, firms, and the government, respectively

When there are two subscripts attached, X_{ij}, interindustry transactions are indicated. The first subscript, i indicates the sector of origin (the provider of inputs), and the second subscript, j, indicates the sector of destination (the user of the inputs). Therefore,

X_{ij} = sales by sector i to sector j, or the value of the inputs of sector i used to produce the output of sector j (i = 1⋯n; j = 1・・・n)

Other key variables are

W_j= wages in sector j (the paymcnts to labor in sector j)

R_j = interest and profits in sectorj (the payments to the owners of capital in sector j)

C_i = personal consumption expenditures on the output of sector i

I_i = investment expenditures for the output of sector i

G_i = govcrnment purchases of the output of sector i

E_i = exports of the output of sector i

M = imports

The rows of the table describe the deliveries of the total amount of a product or primary input to all uses, both intermediate and final. For example, suppose sector 1 represent steel. Then the first row tells us that, out of a gross output of X_1 tons of steel, an amount X_{11} is used in the production of steel itself, an amount X_{12} must be delivered to sector 2, X_{1j} tons are delivered to secot i, X_{1n} to sector n, and F_1 tons are consumed by final end users of steel.

The columns of the table describe the input requirements to produce the gross output totals. Thus, producing the X_1 tons of steel requires X_{11} tons of steel, along with X_{21} units of output from sector 2 (coal, perhaps), X_{i1} from sector i, X_{n1} from sector n, W_1 hours of labor, and R dollars of profits. An entry of 0 in one of the cells of the table indicates that none of the product represented by the row is required by the product represented by the column, so none is delivered.

The n x n matrix in the upper left quadrant of the input-output (l-O) table represents the interindustry transactions or the sales of intermediate goods, X_{ij}, i = 1･･･n, j = 1･･･n. This quadrant describes all the intermediate flows among sectors required to maintain production. The focus is on the interdependent nature of production; each sector ‘s X production depends on the production of the other sectors.

The n x 4 matrix in the upper right quandrant represents the final demands for the output of sector i: by consumers (C_i), firms (I_i), the government (G_i), and foreigners (E_i). It describes the final consumption of produced goods and services, which is more external or exogenous to the industrial sectors that constitute the producers in the economy. Thus it records the sales by each sector to final markets for their production, such as personal consumption purchases and sales to the government, etc,. The demand of these external units which are not used as an input to an industrial production process is generally referred to as final demand.

The 3 x n matrix in the lower left quadrant represents the value added which accounts for the other (nonindustrial) inputs to production. It is composed of the factor payments by each sector to labor (W_j) and the owners of capital (R_j), and payments to foreigners for imports (M_j). All of these inputs (value added and imports) are often lumped together as purchases from what is called the payments sector.

Finally, the lower right quadrant, with relatively few entries, accounts for the final consumption of labor (e.g., domestic help hired by households, W_C, and the employees of the government, W_G), and imports of final goods by consumers (M_C), firms (M_I) and the government (M_G). Thus, the elements in the intersection of the value added row and the final demand column represent payments by final consumers for labor services and for other value added. In the imports row and final demand columns are, for example, M_G which represents government purchases of imported items.

Next, reading across any of the first n rows shows how the output of a sector is allocated across users-as input into the production of the n sectors and for final demands. For example, the totaI demand for the output of sector i, that is, the allocation of the output of the ith sector can be written as

$$X_{ij} = \sum_{j=1}^{n} X_{ij} + F_i \qquad i = 1 \cdots n \tag{1}$$

where $\sum_{j=1}^{n} X_{ij}$ = the total interindustry demand for the output of sector i, or sales by sector i to the n sectors and F_i = the total final demand for the output of sector i.

$F_i = C_i + I_i + G_i + E_i$

Input-Output table can be described mathematically as a set of equations that must be satisfied simultaneously for the gross output of each sector to balance the intermediate and final demand for its product. If you permit each term in equation (1) to represent a cell in the transaction table, then the equation represents row i of the table. There are n equations similar to (1), one for each production sector in the economic system.

Dropping down to the next two rows, we have the total payments to labor and the owners

$$W = \sum_{J=1}^{n} W_j + (W_C + W_G)$$

And

$$R = \sum_{J=1}^{n} R_j$$

The next row indicates the total value of imports into the economy: imports of inputs ($\sum_{j=1}^{n} M_j$) plus imports of final goods and services by consumers, firms, and the government (M_C, M_I, and M_G, respectively).

$$M = \sum_{J=1}^{n} M_j + (M_C + M_I + M_G)$$

Reading down any of the first n column gives the input consumption of the domestic output of a sector. For example, the value of the output of sector j is made up of the value of the domestic inputs purchased from the n sectors plus the value added by domestic labor and capital plus any imported inputs.

Summing down the total output column, total gross output throughout the economy, X_j is

$$X_j = \sum_{i=1}^{n} X_{ij} + W_j + R_j + M_j \; (j = 1 \cdots n) \qquad (2)$$

This same value can be found in (1) by summing across the bottom row; namely $X = (X_1+X_2+X_3 \cdots X_j) + C+I+G+E$

In national income and product accounting, it is the value of total final product that is of interest-goods available for consumption, export, and so on. Equating the two expressions for X and subtracting the common terms from both sides leaves

$$W+R+M = C+I+G+E$$

or $$W+R = C+I+G+(E-M)$$

The left hand side represents gross domestic income-total factor payments in the economy-and the right hand side represent gross domestic product-the total spent on consumption and investment goods, total government purchases, and the total value of net exports from the economy.

Reading down the next four columns gives the value of the final demands by consumers, firms for investment, government purchases, and exports to foreigners.

$$C = \sum_{i=1}^{n} C_i + W_C + M_C$$

$$I = \sum_{i=1}^{n} I_i + M_I$$

$$G = \sum_{I=1}^{n} G_i + W_G + M_G$$

$$E = \sum_{I=1}^{n} E_i$$

By definition, the value of the total demand for the output of any sector (representing the total expenditures) must equal the value of the total supply (indicating the total cost of the output).

Thus, the input-output table can be described mathematically as a set of equations that must be satisfied simultaneously for the gross output of each sector to balance the intermediate and final demand for its product.

We can describe the allocation of the output of kth sector by

$$X_k = \sum_{j=1}^{n} X_{kj} + F_k = \sum_{j=1}^{n} X_{ik} + W_k + R_k + M_k = X_k \ (k = 1 \cdots n) \qquad (3)$$

where

$$F_k = C_k + I_k + G_k + E_k$$

If you permit each term in equation (2) to represent a cell in the input-output table, then the equation represents row i of the table. There are n equations similar to equation (2), one for each production sector in the economic system and, therefore, one equation for each row in the upperquandrants.

6. INPUT-OUTPUT COEFFICIENTS

In input-output work, a fundamental assumption is that the interindustry flows from i to j-recall that these are for a given period, say, a year-depend entirely and exclusively on the total output of sector j for that same time period. Consider the variable that represents intermediate use, X_{ij}. The jth sector Produces some gross output, X_i itself. It uses many intermediate inputs to produce that output, including what it requires from the ith sector, X_{ij}.

Let's define a new number, $a_{ij} = X_{ij} / X_j$. This new number, called a input-output coefficient and this ratio of input to output, X_{ij} / X_j is denoted a_{ij} technical coefficient, can be interpreted as the amount of input i used per unit output of product j, and A complete set of the technical input coefficients of all sectors of a given economy arranged in the form of a rectangular table-corresponding to the input-output table of the same economy-is called the structural matrix of the economy,which in practice are usually computed from input-output tables described in value terms. If we assume a linear production function, we assume that the techinical coefficient is a fixed input requirement for every unit of output by sector j. By definition, we can say

$$X_{ij} = a_{ij} X_j$$

The value of the output of sector j (going down the jth column) can be written as

$$X_j = \sum_{i=1}^{n} X_{ij} + W_j + R_j + M_j \ (j = 1 \cdots n) \tag{3}$$

Dividing through equation (3) by the value of the output of sector j, X_j, we get

$$1 = \sum_{j=1}^{n} X_{ij} / X_j + W_j / X_j + R_j / X_j + M_j / X_j \tag{4}$$

The input-output coefficient, a_{ij}, $0 < a_{ij} < 1$, indicates the share of the output of sector j accounted for by the inputs purchased from sector i. For example, if $a_{13} = 0.15$, then 15% of the value of the output of sector 3 is due to, or contributed by, inputs purchased from sector 1. The input-output coefficients can equal 0, (if no inputs from sector i are used in the production of sector j), but must be less than 1 (if there is value added by labor and capital in the production of the output of sector j).

The W_j / X_j, R_j / X_j, and M_j / X_j, indicate the shareas of wages (payments to labor), interest and profits (payments to the owners of capital), and imports (payments to foreigners) in the output of sector j. Substituting

$a_{ij} = X_{ij} / X_j$ in equation (4), we get

$$1 = \sum_{j=1}^{n} a_{ij} + W_j / X_j + R_j / X_j + M_j / X_j \qquad j = 1 \cdots n \tag{5}$$

We also know that the total demand for the output of sector i is given by

$$X_i = \sum_{j=1}^{n} X_{ij} + F_i \qquad i = 1 \cdots n \tag{6}$$

Substituting $a_{ij}\ X_j = X_{ij}$ (that is, the product of the share of the inputs from sector i in the output of sector j and the output of sectorj must equal the total sales of inputs from sector i to sector j) into equation (6), we get

$$X_i = \sum_{j=1}^{n} a_{ij}X_j + F_i \qquad i = 1 \cdots n \tag{7}$$

Expanding, we have

For

$$\begin{array}{ll}
i = 1: & X_1 = a_{11} X_1 + a_{12} X_2 + \cdots + a_{1n} X_n + F_1 \\
i = 2 & X_2 = a_{21} X_1 + a_{22} X_2 + \cdots + a_{2n} X_n + F_2 \\
\quad . & \quad . \\
\quad . & \quad . \\
i = n: & X_n = a_{n1} X_1 + a_{n2} X_2 + \cdots + a_{nn} X_n + F_n
\end{array}$$

Isolating the final demands on the right-hand side gives:

$$\begin{array}{l}
(1- a_{11}) X_1 - a_{12} X_2 - \cdots - a_{1n} X_n = F_1 \\
-a_{21} X_1 + (1-a_{22}) X_2 - \cdots - a_{2n} X_n = F_2 \\
\qquad . \\
\qquad . \\
-a_{n1} X_1 - a_{n2} X_2 - \cdots + (1 - a_{nn}) X_n = F_n
\end{array} \tag{8}$$

This system of n linear equations in n unknowns (the sectoral outputs,

$X_1, \cdots, X_n$), can be written in matrix notation as (I—A) X = F, where I is the n x n identity matrix, A is the nxn matrix of exogenous input-output coefficients, X is the n x 1 matrix (vector) of endogenous sectoral outputs, and F is the n x 1 matrix (vector) of exogenous final demands.

$$I = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 1 \end{vmatrix}, A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, X = \begin{vmatrix} X_1 \\ X_2 \\ . \\ X_n \end{vmatrix}, \text{ and } F = \begin{vmatrix} F_1 \\ F_2 \\ . \\ F_n \end{vmatrix}$$

The matrix (I—A) is known as the Leontief matrix. The solution to the system, (I—A)X = F, if existing, is found by premultiplying both sides of the equation by the inverse of the Leontief matrix. In this case matrix A satisfies the Hawkins-Simon condition. The matrix $(I—A)^{-1}$ is usually referred to as the multiplier matrix as it shows the direct and indirect requirements of out-put per unit of sectoral final demand.

The inverse matrix, $(I—A)^{-1}$ provides a set of diaaggregated multipliers that are recognized to be more precise and sensitive than Kenesian multipliers (1/1-mpc) for studies of detailed economic impacts. The number 1/ (1-Marginal propensity to onsume)is called the income multiplier in macroeconomics.3/

Leontief inverse matrix, ie., multiplier matrix takes account of the fact that the total effect on output will vary, depending on which sectors are affected by changes in final demand. The total output multiplier for a sector measures the sum of the direct and indirect input requirements from all sectors needed to fulfil the final demand requirements of a given sector. Therefore once the initial change in final demand is known, the values of all inputs and outputs required to supply it can be determined. The basic input-output multiplier we show here is derived from this open input-output model. All components of final demand are treated exogenously. The multiplier repsesents the ratio of the direct and indirect changes to the initial direct changes (in this case, in terms of output) to fulfill the final demand requirements of a given sector.

$$(I-A)^{-1}(I-A)X=(I-A)^{-1}F \tag{9}$$

From such a viewpoint equation (9) can be seen as the result of an iterative process that shows the progressive adjustments of output to final demand and input requirements;

$$X=F+AF+A(AF)+\cdots+A(A^{n-1})F=(I+A+A^{2}+\cdots+A^{n-1})F \tag{10}$$

The first component on the right-hand side of equation (10) shows the direct output requirements to meet the final demand vector F. The second component shows the direct output requirement satisfying, in the second round, the intermediate demand vector, AF needed for the production of vector F in the previous round; the third component shows the direct output requirement for the intermediate consumption, A^2 F, required for the production of vector AF in the previous round, and so on until the process decays and the sum of the series converges to the multiplier matrix $(I-A)^{-1}$.

Thus,

$$\overline{X}=(I-A)^{-1}F=\frac{\mathrm{Adj}\,|I-A|}{|I-A|}F \tag{11}$$

where $\overline{X}$ is the n x 1 matrix (vector) of sectoral output levels required to meet the final demands for the sectoral outputs, given the input requirements set by the sectoral fixed coefficients productions function. The elements of Adjoint (adj (A)) are the cofactor of A.4/ For the solution to exist, the leontief matrix must be nonsingular, that is, $|I-A| \neq 0$. It is quite clear from the above equation that if $|I-A| = 0$, then the inverse would not exist Then, $|I-A|$ has an inverse if and only if $|I-A| \neq 0$ which is tantamount to stating that matrix $|I-A|$ has to be nonsingular. 5/

The adjoint of a matrix A is denoted Adj (A) which is defined only for square matrices and is the transpose of a matrix obtained from the original matrix by replacing its elements a_{ij} by their corresponding cofactors $|C_{ij}|$.

The input-output model of this type makes economic sense only if all of the elements in the vector of gross outputs, $\overline{X}$, are greater than zero and if all of the elements in the vector of final demand, F, are greater than or equal to zero, with at least one element strictly positive. After all, if the gross output of a sector were equal to or less than zero, it would not be producing sector at all. If some element of F were negative, the system would not be self-sustaining; it would require injections to the sector in question from outside.

If all the elements in F were equal to zero, then, according to (9), all elements in X would also be zero. We can be assured that $X_i > 0$ and $F_i \gtrsim 0$, $i = 1, 2, \cdots, n$, and that at least one element of F is greater than zero, if all of principal minors of the matrix $[I_n - A]$, including the determinant of the matrix itself, are strictly greater than zero. This condition is known as the Hawkins-Simon condition, after the economists who first demonstrated it. 6/

7. NUMERICAL EXAMPLE OF AN INPUT-OUTPUT TABLE

An input-output table focuses on the interrelationships between industries in an economy with respect to the production and sese of their products and the products imported from abroad. In a table form (see Figure 2) the economy is viewed with each industry listed across the top as a consuming sector and down the side as a supplying sector.

Now, suppose that we have just three sectors of the economy, for example, agriculture, manufacturing, and services. We have two primary inputs, labor and capital in the value added sector. For simplicity, we will assume a closed economy (no imports or exports), and consider only the level of final demands, F_i, and not the individual components (C_i, I_i, and G_i here). Furthermore, we assume that wage payments by households and the government are zero, so that the lower right quadrant in the input-output table is empty, that is, contains only zero entries. Figure 2 is the hypothetical Input-Output Table valued at producers's prices.

As presented, the input-output table accounts for all the transactions in the economy. Reading across the row for any sector gives the value of the total demand. For example, the total demand for the output of sector 2 of $200 million consists of $65 million in interindustry demands (sectors 1, 2, and 3 use, respectively, $25 million, $20 million, and $20 million worth of output from sector 2 as inputs in their productions) and $135 million in final demands. An example of interindustry demand for intermediate goods would be steel used in manufacturing automobiles. Note that the output of agricultural sector is not used as an input by Services sector, but it is used as an input in its own production.

Figure 2. Illustration of an input-output table (in millions of $)

Purchased by:		Intermediate Users Sectors/Industries			Final Demand	Total Demand
		1	2	3	F	X
Sales by:	1	10	50	0	40	100
Sectors/Industries	2	25	20	20	135	200
	3	5	30	15	10	60
Payments	W	40	60	15		115
	R	20	40	10		70
Total Supply	X	100	200	60		360

Reading down the column for any sector gives the contributios of the inputs to the value of the total output. In this table, the first column represents the vector of gross outputs for the three production sectors and the total amount of labor and capital used as the primary inputs. For example, the $200 million worth of output of sector 2 is accounted for by $50 million, $20 million, and $30 million worth of inputs purchased from secors 1, 2, and 3, respectively, $60 million in payments of wages, and $40 million in payments to owners of capital. Note the sum of the final demands (here $185 million) is equal to the sum of the payments to labor ($115 million) and capital ($70 million).

At this point, we have a set of accounting identities. The contribution of input-output analysis to policy is in assessing the consistency of sectoral output targets, driven by desired growth in final demands, with the expected resource availability.

8. TECHNOLOGY MATRIX A

Input-output analysis became an economic tool when leontief introduced an assumption of fixed-coefficient linear production functions relating inputs used by an industry along each column to its output flow,i.e., for one unit of every industry' output, a fixed amount of input of each kind is required.

The matrix of technical coefficients A will describe the relations a sector has with all other sectors. The matrix of technical coefficients will be matrix such that each column vector represents a different industry and each corresponding vector represents what that industry inputs as a commodity into the column industry. The demand vector will be represented by F. The demand vector F is the amount of product the consumers will need. The total production vector X represents the total production that will be needed to satisfy the demand vector F. The total production vector X will be defined in this section.

To demonstrate with this input-output table, assuming the given technology and fixed coefficients production functions, we can derive the input-output technical coefficients and form the Leontief matrix. We find the matrix of input-output technical coefficients or technology matrix by dividing each column entry by the gross output of the product represented by the column. For example, if X_{11}=\$10 and X_1 =\$100, a_{11}= \$10/\$100= 0.10. Since this is actually \$0.01/\$1, the 0.02 would be interpreted as the "dollar (or Yen)" s worth of inputs from sector 1 per dollar (or Yen)"s worth of output of sector 1". From the equation, $a_{ij} = X_{ij}/X_j$ we get $a_{ij}X_j = X_{ij}$. This is trivial algebra, but it presents the operational form in which the input-output technical coefficients are used.

The input-output technical coefficients, a_{ij} are:

$a_{11} = X_{11}/X_1 = 10/100 = 0.10$, $a_{12} = X_{12}/X_2 = 50/200 = 0.25$, $a_{13} = X_{13}/X_3 = 0/60 = 0.0$

$a_{21} = X_{21}/X_1 = 25/100 = 0.25$, $a_{22} = X_{22}/X_2 = 20/200 = 0.10$, $a_{23} = X_{23}/X_3 = 20/60 = 0.333$

$a_{31} = X_{31}/X_1 = 5/100 = 0.05$, $a_{32} = X_{32}/X_2 = 30/200 = 0.15$, $a_{33} = X_{33}/X_3 = 15/60 = 0.25$

We find that A, the matrix of input-output technical coefficients which is represented below. The matrix below represents the relationships between the industries of Sector 1, Secter 2, and Sector 3. The matrix A is a square matrix, with the same number of rows as of columns as shown below.

$$A = \begin{array}{c} \\ \text{Sector1} \\ \text{Sector2} \\ \text{Sector3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Sector1} & \text{Sector2} & \text{Sector3} \end{array} \\ \left| \begin{array}{ccc} 0.10 & 0.25 & 0.0 \\ 0.25 & 0.10 & 0.333 \\ 0.05 & 0.15 & 0.25 \end{array} \right| \end{array}$$

The principal way in which input-output technical coefficient are used for analysis is as follows. We assume that the numbers in the A matrix represent the structure of production in the economy; the columns are, in effect, the production recipes for each of the sectors, in terms of inputs from all the sectors. To produce one dollar's worth of good 2, for example, one needs as interindustry ingredients 25 cents'worth of good 1, 10 cents'worth of good 2 and 15 cents'worth of good 3. These are, of course, only the inputs needed from other productive sectors; there will be inputs of a more ‘non-produced’ nature as well, such as labor, from the payments sectors.

The relationships between the three industries in example one are as follows.

1. The entry a_{11} holds the number of units sector 1 uses of his own product in producing one more unit of Sector 1. The entry a_{21} holds the number of units the Sector 1 needs of Sector 2 to produce one more unit of Sector 1. The entry a_{31} holds the number of units the Sector 1 needs of Sector 3 to produce onw more unit of Sector 1.
2. The entry a_{12} holds the number of units that the Sector 2 needs from the Sector 1 to produce one more unit of Sector 2. The entry a_{22} holds the number of units the Sector 2 needs of Sector 2 to produce one more unit of Sector 2. The entry a_{32} holds the number of units the Sector 2 needs of Sector 3 to produce one more unit of Sector 2.
3. The entry a_{13} holds the number of units of Sector 1 that Sector 3 needs to produce one more unit of Sector 3. The entry a_{23} holds the number of units of Sector 3 that the Sector 3 needs to produce one more unit of Sector 3. The entry a_{33} holds the number of units of Sector 3 that the Sector 3 needs to produce one more unit of his own product.

In general, each entry in the matrix of input-output technical coefficients or technology matrix is represented as $a_{ij} = x_{ij}/x_j$, where x_j represents the physical output of sector j in our example the total production of an industry. Finally x_{ij} represents the amount of the product of sector i the row industry needed as input to sector j the column industry.

Now let us suppose a input-output technical coefficient or technology matrix as follows;

$$A = \begin{vmatrix} 10/100 & 50/200 & 0/60 \\ 25/100 & 20/200 & 20/60 \\ 5/100 & 30/200 & 15/60 \end{vmatrix}$$

And suppose an external demand vector as follows;

$$F = \begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix}$$

Suppose a total production vector as follows;

$$X = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix}$$

In the argument that follows we will shows that F= X-AX

$$\begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} - \begin{vmatrix} 0.10 & 0.25 & 0.00 \\ 0.25 & 0.10 & 0.333 \\ 0.05 & 0.15 & 0.25 \end{vmatrix} \cdot \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix}$$

To multiply A by X the jth element of X will be multiplied the jth column of A. I have distributed the total production of the Sector 1, the first element in X, through the Sector 1 industry, the first column of A. The total production of the Sector 2 industry. The second element of X. was distributed through the Sector 2 industry, the second column of A. The total production of the Sector 3 industry, the third element of X, was distributed through the Sector 3 industry, the third column of A.

$$\begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} - \begin{vmatrix} 0.10(100) & 0.25(200) & 0.0(60) \\ 0.25(100) & 0.10(200) & 0.333(60) \\ 0.05(100) & 0.15(200) & 0.25(60) \end{vmatrix}$$

Each element of AX is the output of an industry that is used in production. The sector 1 industry produces 10 units for the production needs of itself, 50 units for the production needs of the sector 2 industry and 0 units for the production needs of the Sector 3 industry. The Sector 2 industry produces 25 units for the production needs of the Sector 1 industry, 20 units for its own production needs and 19.98 units for the production needs of the Sector 3 industry. The Sector 3 industry produces 5 units for the production needs of the the Sector 1industry, 30 units for the production needs of the Sector 2 industry and 15 units for the production needs of the Sector 3 industry.

$$\begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} - \begin{vmatrix} 10+50+0 \\ 25+20+19.98 \\ 5+30+15 \end{vmatrix}$$

The total production the Sector 1 industry yields for all industries is 60 units. The total production the Sector 2 industry yields for all industries is 64.98 units. The total production the Sector 3 industry yields for all industries is 50 units.

$$\begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} - \begin{vmatrix} 60 \\ 64.98 \\ 50 \end{vmatrix}$$

Here we see that our demand vector F was in fact equal to our total production minus the production needed by all of the industries.

$$\begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = \begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix}$$

In conclusion, we have shown that F = X—AX so far. Equations are usually written in matrix form, as AX + Y = X, which is the basic input-output system of equations..

9. A little Linear

One of the great advantages of matrix algebra is allowing us to write many linear equations and relationships in a compact way and manipulate them. In other words, matrix notations can serve as a shorthand.

X is the production vector needed to fill both the internal needs and the external demand. We start with D = X—AX. This means that our demand is equal to our total production minus the production needed by other industries as inputs, where total production X is the cumulative product made by each industry whether it is used in production or not. The production needed by other industries as inputs AX is the total amount of product that is used in production.

When making projections for the future you are not given the total production needed. The relations between industries, the input-output technical matrix A, is known and so is the demand for each industry F. Our goal would be to find the total production that will be neded to fill a certain demand. We must solve the equation F= X—AX for X.

Our initial equation from our previous section is

$$F = X - AX.$$

At this point it is necessary to note that there are special type of square matrices, called identity matrices and denoted as I, that consist of 1's on the diagonal that runs from the upper left to the lower right and 0's everywhere else. The 3x3 identity matrix is

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

The virtue of the identity matrices is that the product of an identity matrix with any other matrix for which the product is defined is just the other matrix. In particularly,IX is just X. It turns out that it is often usefrul to represent a matrix as a product with the identity matrix..Thus, any matrix multiplied by an identity matrix is equal to itself IX = X. Therefore we can replace X with IX.

$$F = IX - AX$$

We factor out an X from both terms on the right side of the equation. It is important to factor out the X to the right because if it's factored out to the left matrix multiplication will break down when multiplying the demand vector F on the left side by $(I - A)^{-1}$.

$$F = (I - A) X$$

In order to solve for X we multiply by $(I - A)^{-1}$ on the left side of both sides of the equation.

$$(I-A)^{-1}F = (I-A)^{-1}(I-A)X.$$

any matrix multiplied by it's inverse is equal to the identity matrix $(I-A)^{-1}(I-A) = I$. Substituting I for $(I-A)^{-1}(I-A)$ we get

$$(I-A)^{-1}F = IX$$

Since IX = X as stated before we substitute X for IX, $(I-A)^{-1}F = X$. With a little rearranging we have our equation to solve for the total production needed to satisfy an economy with a known demand vector F and a known input-output technical matrix A.

$$X = (I-A)^{-1}F$$

where V_1, X_2, X_3 are, respectively, outputs of sector 1, sector 2 sector 3. Final demand is given, that is, it is determined outside this model and is called exogenous variable. We are concerned with determining the endogenous variables of the system: outputs of sector1, sector2 and sector 3 in terms of exogenous variables. The purpose of the model is to explain the endogenous variables in terms of the exogenous variables.

Writing it in matrix form we have

$$X = \left[\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0.10 & 0.25 & 0.0 \\ 0.25 & 0.10 & 0.333 \\ 0.05 & 0.15 & 0.25 \end{vmatrix} \right]^{-1} \begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix}$$

We can express the first three rows of the transaction table as

$$\begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} = \begin{vmatrix} 0.10 & 0.25 & 0.0 \\ 0.25 & 0.10 & 0.333 \\ 0.05 & 0.15 & 0.25 \end{vmatrix} \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} + \begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix}$$

Or compactly

$$X = A \text{ x } X + F$$

In order to solve this equation for F, as in equation $(I-A)\ X = F$, we need to subtract A from the third-order identity matrix I_3.

The leontief matrix is $|I-A|$

$$|I_3 - A| = \left[\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0.10 & 0.25 & 0.0 \\ 0.25 & 0.10 & 0.333 \\ 0.05 & 0.15 & 0.25 \end{vmatrix} \right]$$

which equals

$$|I_3 - A| = \begin{vmatrix} 0.90 & -0.25 & 0.0 \\ -0.25 & 0.90 & -0.333 \\ -0.05 & -0.15 & 0.75 \end{vmatrix}$$

10. THE HAWKINS-SIMON CONDITION

Suppose an economy has n industries each producing a single unique product. (There is a generalization of input- output analysis, called activity analysis, in which an industry may produce more than one product, some of which could be pollutants.) Let the product input requirements per unit of product output be expressed as an nxn matrix A. let X be the n dimensional vector of outputs and F the n dimensional vector of final demands. The amounts of production used up in producing output X is AX. This is called intermediary demand. The total demand is thus AX + F. The supply of products is just the vector X.

For an equilibrium between supply and demand the following equations must be satisfied.

$$X = AX + F$$

The equilibrium production is then given by

$$X = (I - A)^{-1} F$$

A viable economy is one in which any vector of nonnegative final demand induces a vector of nonnegative industrial productions. In order for this to be true the elements of $(I - A)^{-1}$ must all be positive. For this to be true $(I - A)$ has to satisfy certain conditions.

A minor of a matrix is the value of a determinant. The principal leading minors of an nxn matrix are evaluated on what is left after the last m rows and columns are deleted, where m runs from $(n - 1)$ down to 0.

The condition for the nxn matrix of $(I - A)$ to have an inverse of nonnegative elements is that its principal leading minors be positive. This is known as the Hawkins-Simon conditions.

We evaluate the naturally ordered principal minors of $|I_3 - A|$ in order to determine whether the Hawkins-Simon condition is met

$$\begin{vmatrix} 0.90 & -0.25 & 0.0 \\ -0.25 & 0.90 & -0.333 \\ -0.05 & -0.15 & 0.75 \end{vmatrix} = 0.51150 > 0$$

$$\begin{vmatrix} 0.90 & -0.25 \\ -0.25 & 0.90 \end{vmatrix} = 0.7475 > 0$$

$$|0.90| = 0.90 > 0$$

From above, we see that the Hawkins-Simon condition is met, because the peincipal minors of $|I_3-A|$ are all positive. Thus, we expect the elements of the final demand vector to be nonnegative with at least one element strictly positive. The vector of final demand, F, equals

$$F = |I_3 - A| X$$

$$= \begin{vmatrix} 0.90 & -0.25 & 0.0 \\ -0.25 & 0.90 & -0.333 \\ -0.05 & -0.15 & 0.75 \end{vmatrix} \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} = \begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix}$$ 6/

In order to solve this equation for X, we must find the multiplicative inverse of $|I_3 - A|$. The inverse of the Leontief matrix is $|I-A|^{-1}$.7/

The inverse of $|I-A|$ =

$$C^T / |I-A| = \frac{1}{|D|} \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}^T = \begin{vmatrix} 0.6250 & 0.2041 & 0.0825 \\ 0.1875 & 0.6750 & 0.1475 \\ 0.0832 & 0.2997 & 0.7475 \end{vmatrix}^T$$

$$|D| = 0.9\begin{vmatrix} 0.90 & -0.333 \\ -0.15 & 0.75 \end{vmatrix} + 0.25\begin{vmatrix} -0.25 & -0.333 \\ -0.05 & 0.75 \end{vmatrix} = 0.90(0.675\text{-}04995)8+$$

$$0.25\,(-0.1875-0.01665) = 0.5115075 \quad 9/$$

$$|I_3 - A|^{-1} = \frac{1}{0.5115}\begin{vmatrix} 0.6250 & 0.1875 & 0.0832 \\ 0.2041 & 0.6750 & 0.2997 \\ 0.0825 & 0.1475 & 0.7475 \end{vmatrix} = \begin{vmatrix} 1.2218 & 0.3665 & 0.1626 \\ 0.3990 & 1.3196 & 0.5859 \\ 0.1612 & 0.2883 & 1.4613 \end{vmatrix}$$

Note that the main diagonal elements in the inverse of the Leontief matrix are all greater than 1, while the off-diagonal elements are positive and less than 1.10/

Now the system of linear equations, representing the input-output table, can be written as

$$\bar{X} = \begin{vmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{vmatrix} = \begin{vmatrix} 100 \\ 200 \\ 60 \end{vmatrix} = \begin{vmatrix} 1.2218 & 0.3665 & 0.1626 \\ 0.3990 & 1.3196 & 0.5859 \\ 0.1612 & 0.2883 & 1.4613 \end{vmatrix} \begin{vmatrix} 40 \\ 135 \\ 10 \end{vmatrix} = |I - A|^{-1}F$$

$\bar{X}_1 = 1.2218 \text{ x } 40 + 0.3665 \text{ x } 135 + 0.1626 \text{ x } 10 \quad 100$

$\bar{X}_2 = 0.3990 \text{ x } 40 + 1.3196 \text{ x } 135 + 0.5859 \text{ x } 10 \quad 200$

$\bar{X}_3 = 0.1612 \text{ x } 40 + 0.2883 \text{ x } 135 + 1.4613 \text{ x } 10 = 60$

All that we have accomplished so far, in equations above, is to illustrate that the system is balanced. That is, the gross output of each sector does satisfy the intermediate and final demands.

To illustrate the use of the input-output model for consistency planning, suppose the goals over the planning horizon were to increase the sectoral outputs enough to produce a new vector of final demands of F = (60 140 20). Suppose further that the availability labor was projected to be \$140 million and the available capital to be \$80 million. Recall that we are assuming constant returns to scale and fixed unit prices for inputs and outputs. The question is whether the projected labor and capital will be sufficient for producing sectoral outputs consistent with the desired new levels of final demands.

We can use the equation below to determine the full effects of the change on the gross output of every sector. We simply premultiply the new vector of final demand, F'by $|I-A|^{-1}$ to find the new vector of gross outputs, $\overline{X}$'.

The new level of sectoral outputs, $\overline{X}$, is given by $\overline{X}=|I-A|^{-1}F'$.

$$\overline{X}' = \begin{vmatrix} 127.87 \\ 220.40 \\ 79.26 \end{vmatrix} = \begin{vmatrix} 1.2218 & 0.3665 & 0.1626 \\ 0.3990 & 1.3196 & 0.5859 \\ 0.1612 & 0.2883 & 1.4613 \end{vmatrix} \begin{vmatrix} 60 \\ 140 \\ 20 \end{vmatrix}$$

That is, we simply rotate in the vector of desired final demands, F', and solve for the new secoral output levels. Summarizing, to meet the desired increases of \$20 million, \$5million, and \$10 million in the final demands for the outputs of secors 1, 2, and 3, respectively, requires increases of \$27.91 million, \$20.41 million, and \$19.28 million in the total sectoral outputs produced.

The preceding analysis shows us how to balance gross output with intermediate and final demands, but it does not tell us whether the increased output levels are feasible. To determine the feasibility of the change, we must know whether we have sufficient primary inputs to sustain the higher gross output amounts. Consider the bottom two rows of the input-output table in Figure 2. We can find input-output technical coefficients for labor and capital by dividing the column entities by the gross output of the product represented by the column. The required inputs of labor and capital can be found using the fixed shares of wages and payments of interest and profits in the total values of the sectoral outputs with the new levels of sectoral outputs.

Returning to Figure 2 and the input-output table, the shares of wages (W) and interest and profits (R) in the value of sectoral outputs are

$$W_1/X_1 = 40/100 = 0.4,\ W_2/X_2 = 60/200 = 0.3,\ \ W_3/X_3 = 15/60 = 0.25$$

$$R_1/X_1 = 20/100 = 0.2,\ R_2/X_2 = 40/200 = 0.25,\ R_3/X_3 = 10/60 = 0.167$$

The new total wage payments required are

$$[W_1/X_1\ W_2/X_2\ W_3/X_3]\begin{vmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \end{vmatrix} = [0.4\ 0.3\ 0.25]\begin{vmatrix} 127.91 \\ 220.41 \\ 79.28 \end{vmatrix} = [51.16+66.12+19.82]$$

$$= 137.1$$

With the available labor projected to be \$140 million, it appears that there would be more than enough labor to produce the required new sectoral outputs. Indeed, with the fixed technical coefficients production functions, the surplus labor of \$2.9 million (\$140－\$137.1) would be unemplyed.

The new total payments of interest and profits are

$$[R1/X1\ R2/X2\ R3/X3] \begin{vmatrix} \overline{X}1 \\ \overline{X}2 \\ \overline{X}3 \end{vmatrix} = [0.2\ 0.2\ .0167] \begin{vmatrix} 127.91 \\ 220.41 \\ 79.28 \end{vmatrix} = [25.58+44.08+13.24]$$

$$= 82.9$$

In contrast, the required payments to the owners of capital of $82.9 million would exceed the capital projected to be available (assumed to be $80 million). Therefore, the new levels of sectoral outputs required to meet the desired new levels of final demands are inconsistent with the projected amount of capital to be available. Either the desired levels of final demands must be pared back or measures taken to increase the amount of pysical capital formation, or some combination of these two approaches.

With no input substitution allowed with the fixed coefficients production functions, the option of substituting some of the surplus labor for the scarce capital, that is, switching to a more labor-intensive method of production, is not possible.

Now we have a completely new input-output table, consistent with the new final demand vector, F. We find the total amounts of labor and capital necessary to produce the new output levels by summing the intermediate use and final demand entries of each row. The completely new Input Output table is presented in Figure 3.

Figure3. Revised Input-0utput table for New Final Demand Vector
(in millions of $)

Purchases by:		Intermediate Users Sectors/Industries			Final Demand	Total Demand
Sectors/Industries		1	2	3	F	X
Sales by:	1	12.77	55.10	0	60	127.87
Sectors/Sectors/Industries	2	31.97	22.03	26.40	140	220.40
	3	6.39	33.06	19.80	20	79.26
Payments	W	51.16	66.12	19.82		137.1
	R	25.58	44.08	13.24		82.9
Total Supply	X	127.87	220.40	79.26		427.53

After all, the tables and equations of input-output analysis represent nothing more than a description of a balance between gross output and total demand for all sectors of an economic system. They take into account intersectoral flows-the fact that sector j depends on the outputs of itself and the other sectors to produce its output.

11. COMPARATIVE STATICS

We conclude our discussion of input-output analysis with an illustration of comparative statics. In the earlier example, we introduce a vector of new final demands and solved for the required new sectoral output levels. From the general solution to the input-output model, however, we can isolate the impact of a change in any final demand on the output of any sector in the economy.

Return to the general solution from the input-output table, $\overline{X} = (I - A)^{-1} F$, which we can write as

$$\begin{vmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_n \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nn} \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ F_n \end{vmatrix}$$

where b_{ij}, ($i = 1 , , , n; j = 1 , , , n$), is the (i, j)the element of the inverse of the Leontief matrix, $(I-A)^{-1}$. Expanding, we have

$$\bar{X}_1 = b_{11}F_1 + b_{12}F_2 + \cdots + b_{1n}F_n$$
$$\bar{X}_2 = b_{21}F_1 + b_{22}F_2 + \cdots + b_{2n}F_n$$
$$\cdot$$
$$\cdot$$
$$\bar{X}_n = b_{n1}F_1 + b_{n2}F_2 + \cdots + b_{nn}F_n$$

Or, in general,

$$\bar{X}_i = b_{i1}F_1 + b_{i2}F_2 + \cdots + b_{in}F_n$$

To find the effect of a change in the final demand for the output of sector j on the required equilibrium output of sector i, we take the partial derivative,

$$\delta\bar{X}_i \ / \ \delta F_j = b_{ij}.$$

Recall the earlier example of the three-sector economy. In general, the equilibrium solution to the I-O table of Figure 1 can be written as

$$\bar{X} = \begin{vmatrix} \bar{X}_1 \\ X_2 \\ X_3 \end{vmatrix} = \begin{vmatrix} 1.2218 & 0.3665 & 0.1626 \\ 0.3990 & 1.3196 & 0.5859 \\ 0.1612 & 0.2883 & 1.4613 \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix}$$

To show this more clearly, we can expand the system into the three linear equations:

$\overline{X}_1 = 1.2218\ F_1 + 0.3665\ F_2 + 0.1626\ F_3$

$\overline{X}_2 = 0.3990\ F_1 + 1.3196\ F_2 + 0.5859\ F_3$

$\overline{X}_3 = 0.1612\ F_1 + 0.2883\ F_2 + 1.4613\ F_3$

To find the effect of a \$1million increase in the final demand for the output of sector 3 on the equilibrium outputs of sectors 1, 2, and 3, respectively, we take the following partial derivatives:

$$\delta\, \overline{X}_1 \ / \ \delta F_3 = b_{13} = 0.1626$$

Ceteris paribus, a \$1million increase (decrease) in the final demand for the output of sector 3 requires an increase (a decrease) of \$.1026 million in the equilibrium output of sector 1.

$$\delta\, \overline{X}_2 \ / \ \delta F_3 = b_{23} = 0.5859$$

Ceteris paribus, a \$1million increae (decrease) in the final demand for the output of sector 3 requires an increase (a decrease) of \$.5859 million in the equilibrium output of sector 2.

$$\delta\, \overline{X}_3 \ / \ \delta F_3 = b_{33} = 1.4613$$

Ceteris paribus, a \$1 million increase (decrease) in the final demand for the output of sector 3 reqires an increase (a decrease) of \$1.4613 million in the equilibrium output of sector 3. That is, an increse in the final demand for the output of sector 3 by \$1 million would increase the output of sector 3 not only by this \$1million, but by an additional \$.4613 million to meet the intermediate demands from sectors 1, 2, and 3 for inputs from sector 3 as the outputs of these sectors are increased in response to the intermediate demands from secto 3 for inputs.

In an input-output model with n sectors, there are n^2 such comparative static results, given by the n^2 elements in the invese of the Leontief matrix.11/

The input-output analysis we have discussed here is based on “an Open input-output model which contains an open sector which exogeneously determines a final demand for the product of each industry and supplies a primary input not produced by the n industries themselves. If the exogenous sector of the open input-output model is absorbed into the system as just another industry, the model will become a closed model. 12/

EXERCISE Ⅰ: Suppose you are in charge of economic planning for a region characterized by three production sectors; grain production, automobiles, and electrical power. Last year, the grain sector consumed 3 units of its gross output in its own production process and delivered 5 units to automobiles and 10 units to final consumers. The automobile sector delivered 4 units to grian, 2 units to electrical power, and 6 units to final consumers, and it used 2 units in its own production. Electrical power used 3 units of elecricity in its own production, and it delivered 20 units to automobile, 5 units to grain, and 8 units to final consumers. Your region has one primary input, labor, which supplied 15 units to grin, 10 units to automobiles, and 5 units to electrical power. In addtion, 4 units of labor were employed by final consumers.

1. Set up the input-output table for this economic region.

Purchases by		Intermediate Users Sectors/Industries			Final Demand	Total Demand
		1	2	3	F	X
Sales by:	1	3	5	0	10	18
Sectors / Industries	2	4	2	2	6	14
	3	5	20	3	8	36
Payments Labor		15	10	5	4	34
Total supply X		18	14	36		68

2. Express quadrants I and II of your table as a matrix of equations, with a vector of gross output, X, a matrix of technical coefficients, A, and a vector of final demand, F.

$$X = AX + F = \begin{vmatrix} 18 \\ 14 \\ 36 \end{vmatrix} = \begin{vmatrix} 0.167 & 0.357 & 0 \\ 0.222 & 0.143 & 0.056 \\ 0.278 & 1.429 & 0.083 \end{vmatrix} \begin{vmatrix} 18 \\ 14 \\ 36 \end{vmatrix} + \begin{vmatrix} 10 \\ 6 \\ 8 \end{vmatrix}$$

3. Solve your matrix equation for F, and then solve it for X.

$$F = \left| I - A \right| X = \begin{vmatrix} 10 \\ 6 \\ 8 \end{vmatrix} = \begin{vmatrix} 0.833 & -0.357 & 0 \\ -0.222 & 0.857 & -0.056 \\ -0.278 & -1.429 & 0.917 \end{vmatrix} \begin{vmatrix} 18 \\ 14 \\ 36 \end{vmatrix}$$

$$X = \left| I - A \right|^{-1} F = \begin{vmatrix} 18 \\ 14 \\ 36 \end{vmatrix} = \begin{vmatrix} 1.38 & 0.641 & 0.039 \\ 0.429 & 1.497 & 0.091 \\ 1.089 & 2.527 & 1.244 \end{vmatrix} \begin{vmatrix} 10 \\ 6 \\ 8 \end{vmatrix}$$

4. Describe the impact of a 3-unit decrease in the gross output of the grain sector on all sectors of your region. Construct the input-output table for the new situation.

Purchases by		Intermediate Users Sectors/Industries			Final Demand	Total Demand
		1	2	3	F	X
Sales by:	1	2.5	5	0	7.5	15
Sectors/Industries	2	3.3	2	2	6.7	14
	3	4.167	20	3	8.83	36
Payments Labor		12.5	10.0	5	4	31.5
Total supply X		15.0	14.0	36		65.0

5. Describe the impact of an increase in the final demand for automobiles of 2 units (assume the original input-output table from question 1 to begin with)

Construct the input-output table for new situation.

Purchases by		Intermediate Users Sectors/Industries			Final Demand	Total Demand
		1	2	3	F	X
Sales by:	1	3.21	6.07	0	10	19.28
Sectors/Industries	2	4.29	2.43	.28	8	17.0
	3	5.36	24.28	3.42	8	41.05
Payments Labor		16.07	12.14	.70	4	37.9
Total supply X		19.28	17.0	.05		77.33

EXERCISE Ⅱ: The following is an input-Output table in the case of competitive Imports. Competitive imports can be represented in a technical coefficient matrix, while noncompetitive imports cannot. Competitive imports are usually handled by adding transactions to the domestic transactions matrix as if they were domestically produced. Derive and present the basic input-output model with competitive imports and illustrate the input-output calculations step by step using fugures in the table.

We introduce two fundamental assumptions, one for input coefficients and the other for import coefficients.
Therefore,

$X = [I - (I - \hat{M})A]^{-1}(I - M)F$ (Domestic demand) + E (Exports)]. This shows that domestic final demand (F) and export (E) produces domestic production (X). Here $(I - \hat{M})A$ indicates the import ratio of domestic products when the import input ratio is assumed to be constant in all sectors, regardless of whether they are for intermediate demand or final demand. $(I - \hat{M})F$ indicates domestic final demand for domestic products under the same assumption. This is because in this competitive imports model, import ratios for individual items (for rows) (or import coefficients) are assumed to be identical in all output sectors. We also assume that the matrix $[I - (I - \hat{M})A]$ is nonsingular.

Input-Output Table at producers' prices

Output / Input	Intermediate Demand			Final Demand				Gross Output
	Sector1	Sector2	Sector3	C	I	E	IM	
Sector1	20	75	35	130	0	40	−100	200
Sector2	30	80	95	330	0	25	−60	500
Sector3	60	45	100	55	190	65	−215	300
Valueadded	90	300	70					
Gross Output	200	500	300					

This table shows the Competitive Imports Model, clealy indicating imports. For row items, both intermediate demand (X_{ij}) and final demand (F_i) are supplies including imports, and columns and rows (productions) offset each other because imports are indicated negative values.

The input-output technical coefficient are calculated, as a matter of simple arithmetic, by dividing the elements in the intermediate product vectors by the corresponding output total. Thus, the matrix of coefficient is given by

$$[I-(I-\hat{M})\,A] = \begin{vmatrix} 0.9384 & -0.0929 & -0.0715 \\ -0.1331 & 0.8579 & -0.2810 \\ -0.1566 & -0.0470 & 0.8259 \end{vmatrix}$$

The inverse matrix referred to as the Leontief inverse is given by

$$[I-(I-\hat{M})\,A]^{-1} = \begin{vmatrix} 1.107 & 0.127 & 0.139 \\ 0.245 & 1.216 & 0.435 \\ 0.224 & 0.094 & 1.263 \end{vmatrix}$$

$\hat{M}$ (rates of imports) = M / (AX + C + I) =

$$\begin{vmatrix} 0.3849 & 0 & 0 \\ 0 & 0.1121 & 0 \\ 0 & 0 & 0.4771 \end{vmatrix}$$

Since import coefficients by row can be defined as follows;

$$\hat{M}_i = M_i \,/\, \Sigma a_{ij}X_j + F_i$$

M_i represents the ratio of imports in product“i”within total domestic demands, or ratios of dependence on imports whilel $(I-\hat{M}_i)$ represents self-sufficiency ratios.

The diagonal matrix (M) can be assumed to have an import coefficient (M) as the diagonal element and zero as non-diagonal elements.

$$(I-\hat{M}_i) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1.3846 & 0 & 0 \\ 0 & 0.1121 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0.6154 & 0 & 0 \\ 0 & 0.888 & 0 \\ 0 & 0 & 0.5223 \end{vmatrix}$$

The basic input-output model is shown below;

$$X \qquad [I-(I-\hat{M})\,A]^{-1} \qquad [(1-\hat{M})\,(C+I)+E]$$

$$\begin{vmatrix} 200 \\ 500 \\ 300 \end{vmatrix} = \begin{vmatrix} 1.107 & 0.127 & 0.139 \\ 0.245 & 1.216 & 0.435 \\ 0.224 & 0.094 & 1.263 \end{vmatrix} \times \begin{vmatrix} 120 \\ 318 \\ 192.93 \end{vmatrix}$$

In addition, we add the inverse matrix coefficients table below;

Inverse matrix coefficient table

Output / Input	Sector 1	Sector 2	Sector 3	Sum of Row	Response ratios
Sector 1	1.107	0.127	0.139	1.373	0.849
Sector 2	0.245	1.216	0.435	1.896	1.172
Sector 3	0.224	0.094	1.263	1.581	0.977
Sum of column	1.576	1.437	1.837	4.850	
Effect ratios	0.974	0.888	1.136		

This inverse matrix coefficient table indicates how production will be ultimately induced in what industry or sector by a demand increase of one unit in a certain industry or sector.

The figure in each column in the inverse matrix coefficients table indicates the production required directly and indirectly at each row sector when the final demand for the column sector (that is, demand for domestic production) increases by one unit. The total (sum of column) indicates the scale of production repercussions on entire industries or sectors, caused by one unit of final demand for the column sector.

Column sum =1.576+1.437+1.837=4.85 and dividing 4.85 by 3 (sum of sectors) makes the mean value of entire vertical sum in inverse matrix coefficient table, i.e.,1.6166. Then again, we divide each column sum by the mean value, 1.6166, we get the effect ratios of every column sector as following:

Sector1: 0.974, Sector2: 0.888, Sector3:1.136

The vertical sum of every column sector of the inverse matrix coefficients is divided by the mean value of the entire sum of column to produce a ratio. This ratio indicates the relative magnitudes of production repercussions, that is, which sector's final demand can exert the greatest production repercussions on entire sectors. This is called an effect ratio. In this case, sector 3 has relatively high value, indicating that sector3 exert great production repercussions on entire industries or sectors.

The figure for each row in the inverse matrix coefficient table indicates the supplies required directly and indirectly at each row sector when one unit of the final demand for the column sector at the top of the table occurs. The ratio produced by dividing the total (horizontal sum) by the mean value of the entire sum of row will indicate the relative influences of one unit of final demand for a row sector, which can exert the greatest production repercussions on entire industries or sectors. This is called a response ratio.

Row sum =1.373+1.896+1.581=4.850 and dividing 4.850 by 3 makes the mean value of the entire horizontal sum in inverse matrix coefficients of

1.6166. We divide each sum of row in inverse matrix coefficient table by the mean value of the entire horizontal sum and then, the following figures;

1.373/1.6166=0.849. 1.896/1.6166=1.172. 1.581/1.6166=0.977.

By combining these two of the effect ratios and the response ratios, we can create a typological presentation of the functions of each industrial sector.

Footnote:

1/The physiocrats divided society into three classes or sectors. First, a productive class of cultivators engaged in agricultural production were solely responsible for the generation of society'surplus product, a part of which formed net investment. Second, the strile class referred to producers of manufactured commodities. The term sterile was applied not because manufacturers did not produce anything of value, but because the value of their output (e.g., clothes, shoes, cooking utensils) was presumed to be equal to the necessary costs of raw materials received from the cultivators plus the subsistence level of the producer wages. According to the Physiocrats no surplus product or profits were thought to originate in manufacturing. Lastly, came landlords or the idle class who through the money they received as rent consumed the surplus product created by the productive cultivators. Of particular relevance to the contemporary method of input-output analysis is the Tableau's lengthy depiction of the three classes' transactions. Once the landlord class received their money rents, account was made of the transactions that lead to distribution of products between the agricultural and manufacturing sectors. In short, the Tableau illustrated the two sectors'interdependence as the output from each sector served as a necessary input for other. These are exactly the type of interindustry relationships that form the core theoretical foundation upon which modern day input-output analysis rests.

2/Imports in an input-output framework are usually divided into basic groups (1) imports of commodities that are also domestically produced (competitive imports) and (2) imports of commodities that are not domestically produced (noncompetitive imports). The distinction is that competitive imports can be represented in a technical coefficients matrix, while noncompetitive imports cannot. Competitive imports are usually handled by adding transactions to the domestic transactions matrix as if they were domestically produced. This treatment of competitive imports has been adopted in input-output studies in Japan. thus, the inverse matrix

coefficients in the $[I-(I-\hat{M})A]^{-1}$ Type are commonly utilized.
Input coefficients include imports. This implies that all repercussions derived from final demand do not necessarily induce domestic production, some effects may induce imports. See EX.II..
3/The income multiplier shows the overall total of direct and indirect effect of a dollar increase in final demand. Multiplier analysis is carried out strictly at the macro level. it does not ask who will produce the extra output when final demand is increased, or in which sector of the economy. The additional national product is used. This shortcoming of macroanalysis can be eliminated if input-output method is used instead..

4/The adjoint matrix of a square matrix:
1. The minor of an element is the determinant of what remains when the row and column containing that element is crossed out
2. The cofactor of an element is the value of the minor multiplied by ±1. The ± is given by the determinant of signs
3. Given a matrix A, the inverse of A is defined as follows:

$$A-1 = \frac{(C)\,T}{|A|} \qquad A = \begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$$

where CT is the matrix in which every element is replaced by its cofactor:

$$A^{-1} = \frac{1}{|A|} \begin{vmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{vmatrix}^{T}$$

$$\mathrm{Adj}\,(A) = |C|^{T} = \begin{vmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{vmatrix}$$

where the elements of the adjoint are the cofactors of A. For example,

$$|C_{23}| = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}a_{32} - a_{12}a_{31})$$

The adjoint of a matrix A,denoted adj (A) is defined only for square matrices and is the transpose of a matrix obtained from thr original matrix by replacing its elements aij by their corresponding cofactors C_{ij}.

5/The inverse matrix:

The minor of an element is the determinant of what is left, when the row and column containing that element are crossed out. For example, in determinant

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

The coefficient matrix is nonsingular ($D = -13$) so that the inverse matrix exists.

The minor of the first element in the first row, $d_{1,1}$, is

$$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = (2)\,(2) - (3)\,(3) = -5$$

The cofactor of a given element is the minor of that element multiplied by either +1 or -1. If the given element is in the same position as +1 in the determinant of +1 signs,given below, multiply theminoe by (+1). Otherwise, multiply the minor by a (-1).

$$\text{Determinant of signs:} \quad \begin{vmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{vmatrix}$$

The cofactor of an element is refered to by capital C, subscripted with the location (row, column) of the element. For example, C_{11} is the cofactor of $d_{1,1}$ in determinant D above. It is calculated as follwos;

$$C_{11} = (\text{minor of } d_{1,1}) \times (+1) = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \times (1) = (-5)\,(1) = -5$$

(the minor= -5, was evaluated above)

The value of a 3x3 determinant is calculted as follows;

$$|D| = (d_{1,1} \times C_{1,1}) + (d_{1,2} \times C_{1,2}) + (d_{1,3} \times C_{1,3})$$

That is, the value of $|D|$ is the sum of the products of each element in row 1 and its cofactor (in fact, $|D|$ may also be evaluated by summing the products of each element x cofactor from any one row or column. This is particularly useful of a row or column contains several zero.) This method of evaluation is called Laplace expansion or the cofactor method.

Given a matrix A, the inverse of A is defined as follows;

$$A^{-1} = \frac{(C)^T}{|A|}$$

Where $|C|^T$ is the matrix in which every element is replaced by its cofactor.

$$A^{-1} = \frac{1}{|A|}\begin{vmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{vmatrix}^T$$

6/ To understand the Hawkins-Simon conditions, it is worthwhile examining a two sector case graphically. As we know, with two sectors, the condition $(I - A)X=F$ can be written:

$$\begin{matrix} (1- a_{11}) & -a_{12} \\ -a_{21} & (1- a_{22}) \end{matrix} \begin{matrix} X_1 = F_1 \\ X_2 = F_2 \end{matrix}$$

Or,

$$(1-a_{11})\,X_1 \quad -a_{12}X_2 \quad = F_1 \quad (a) \quad 1\text{-}2$$
$$-a_{21}\,X_1 \quad + (1-a_{22})\,X_{22} = F_2 \quad (b)$$

Thus, we have two lenear equations. Given F_1 and F_2, we can draw two lines in a (X_1, X_2) space (denoted L_1 and L_2) for the system of equations as in Figure 1. Frontier L_1 maps the levels of X_1 and X_2 that satisfy the first equation and L_2 maps the levels which satisfy the second equation. As per the first equation, line L_1 has vertical intercept $F_1/(-a_{12}) < 0$, horizontal intercept $F_1/(1-a_{11}) > 0$ and slope $(1-a_{11})/a_{12}$. From the second equation, line L_2 has vertical intercept $F_2/(1-a_{22}) > 0$, horizontal intercept $F_2/(-a_{21}) < 0$ and slope $a_{21}/(1-a_{22}) > 0$. Thus, the equilibrium values of X_1 and X_2 which satisfy both equations must be at the intersection of the two loci L_1 and L_2. This is shown in Figure 1 as the points (X_1^*, X_2^*).

As long as $(1-a_{11}) > 0$ and $(1-a_{22}) > 0$—the first Hawkins-Simon condition in the 2x2 casefor $F_1 > 0$ and $F_2 > 0$, the intercept of Eq (1−2) (a) on the X_1−axis will be to the right of the origin and the intercept of Eq. (1−2) (b) on the X_2-axis will be above the origin. Therefore, for nonnegative total outputs, it is required that these two equations intersect in the first quadrant, which means that the slope of equation (a) must be greater than the slope of equation (b).

It is easy to notice that an intersection is guaranteed only if the slope of L_1 is greater than the slope of L_2,

These slopes are: For equation (a) $L_1 = (1-a_{11}) / a_{12}$

For equation (b) $L_2 = a_{21} / (1-a_{22})$

And thus the slope requirement is $(1-a_{11}) / a_{12} > a_{21} / (1-a_{22})$. Multiplying both sides of the inequality by $(1-a_{22})$ and by a_{12} − both of which are assumed to be strictly positive-does not alter the direction of the inequality, giving $(1-a_{11})(1-a_{22}) > a_{12}a_{21}$

or

$(1-a_{11})(1-a_{22}) - a_{12}a_{21} > 0$, which is just $|I-A| > 0$, the second Hawkins-Simon condition in the 2x2 case and which, it must be noticed, merely states that the determinant of the matrix $(1-A)$ is positive. This is precisely the Hawkins-Simon condition applied to the two-sector case, If, on the other hand, $|I-A| < 0$, then notice that this would imply that $(I-a_{11}) / a_{12} < a_{21} / (I-a_{22})$ so that the slope of L_1 would be smaller than the slope of L_2 which, as we can immediately see diagrammatically, implies that L_1 and L_2 will not intersect-i.e. there is no non-negative solution X_1^*, X_2^*.

L_1 $X_2 = ((1-a_{11}) / a_{12}) X_1 - (1/a_{12})F_1$

L_2 $X_2 = (a_{21}/(1-a_{22})) X_1 + (1/(1-a_{22}))F_2$

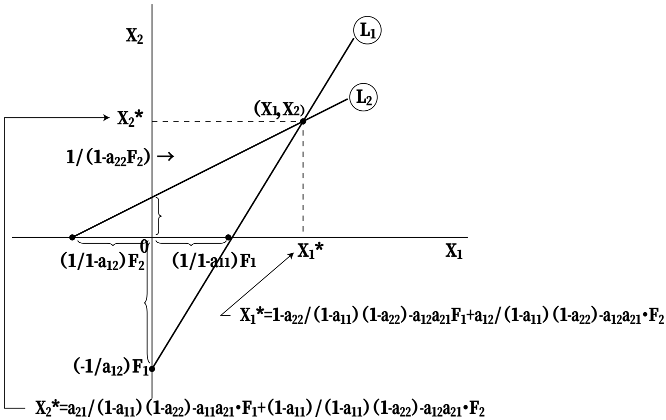

Figure 1 Quantity Determination

7/Leontief inverse of the input-output coefficientsx

If the number of sectors,n is large it can be aukward obtaining the inverse of the leontief matrix. In these circumstances it may be convenient to use the fact that

$$\begin{aligned}
&(I - A)(I + A + A^2 + A^3 + A^4 + A^5 + \cdots + A^m)\\
&= I(I + A + \cdots + A^m) - A(I + A + \cdots + A^m)\\
&= (I + A + \cdots + A^m) - (A + A^2 + \cdots + A^{m+1})\\
&= I - A^{m+1}
\end{aligned}$$

Now if the product had resulted in the unit matrix then the expression $(A + A^2 + \cdots + A^m)$ would be the inverse of $I-A$. But the presence of A^{n+1} prevents this.

However, since every element of A is positive and less than 1— and, more importantly, the total of each column of A adds up to less than 1 — then as m increases, A^{m+1} tends to the null matrix. Therefore, approximately,

$(I - A)^{-1} = I + A^2 + A^3 + A^4 + A^5 + \cdots + A^m$ as n approaches infinity and the size of the last term, A^m, is a guide to how close the approximation is. The inverse matrix (I — A) is fundamental to input-output analysis as it shows the full impact of an exogenous increase in net full demand on all industries. With such a matrix it is possible to unravel the technological interdependence of the productive system and trace the generation of output of demand from final consumption which is part of net final demand throughout the system. This Leontief inverse is always non-negative when:

1)A is measured in value terms;

2}a_{ij}, any element of A is non-negative and smaller than 1, which under normal economic conditions, is always satisfied since the value of any input used is smaller than the value of output.

8/ Students often assimilate matrix with determinant and vice versa. A matrix is used to separate the individual elements of a set. A determinaant is, on the other hand, a number. However, some matrices do have determinants. All aquare matrices have determinants. In matrix algebra square matrices are of special importance. The determimant of a square matrix A is

$|A| = |a_{ij}|$.

A matrix is shown by capital letters and determinnants by light capital letters.

The elements of a determinant are formed in the same manner as the elements of its matrix, and the value of determinant is indicated by a pair of vertical lines placed on the side of the matrix. Determinant are devices which will lead us to the results of the system of linear equation from which we have gathered the matrix.

9/ Do not be concerned by small rounding errors.

10/ To calculate the determinant of

$$A = \begin{vmatrix} 1 & -4 & 2 & -2 \\ 4 & 7 & -3 & 5 \\ 3 & 0 & 8 & 0 \\ -5 & -1 & 6 & 9 \end{vmatrix}$$

We expand along the tird row, because it it the row or column containing the most zeros.

$$\text{Det } A = 3A_{31} + 0A_{32} + 8A_{33} + 0A_{34} = 3\,(1)^{3+1}\,M_{31} + 8\,(-1)^{3+3}\,M_{33}$$

$$= -4\,(1)\,(-57) + 2\,(-1)\,(68) + (-2)\,(1)\,(39) = 14$$

And

$$M_{31} = \begin{vmatrix} 1 & -4 & -2 \\ 4 & 7 & 5 \\ -5 & -1 & 9 \end{vmatrix} = 1\,(-1)^2 \begin{vmatrix} 7 & 5 \\ -1 & 9 \end{vmatrix} + (-4)\,(-1)^3 \begin{vmatrix} 4 & 5 \\ -3 & 9 \end{vmatrix} + (-2)\,(-1)^4 \begin{vmatrix} 4 & 7 \\ -5 & -1 \end{vmatrix}$$

$$= 1\,(1)\,(68) + (-4)\,(-1)\,(61) + (-2)\,(1)\,(31) = 250$$

Then det A = 3 (1) (14) + 8 (1) (250) = 2042

11/ To determine the inverse of

$$A = \begin{vmatrix} 0 & 1 & 1 \\ 5 & 1 & -1 \\ 2 & -3 & -3 \end{vmatrix}$$

There are a series of row operations that transform matrix A into an identity matrix. If we do the same exact transformations, and in the same order, on an identity matrix, we transform it into A^{-1}.

$$[A \mid I] = \begin{vmatrix} 0 & 1 & 1 & \cdot\, 1 & 0 & 0 \\ 5 & 1 & -1 & \cdot\, 0 & 1 & 0 \\ 2 & -2 & -3 & \cdot\, 0 & 0 & 1 \end{vmatrix} \begin{matrix} -> \\ -> \\ \end{matrix} \begin{vmatrix} 5 & 1 & -1 & \cdot\, 0 & 1 & 0 \\ 0 & 1 & 1 & \cdot\, 1 & 0 & 0 \\ 2 & -3 & -3 & \cdot\, 0 & 0 & 1 \end{vmatrix}$$ Interchanging the first and second row

$$-> \begin{vmatrix} 1 & 1/5 & -1/5 & \cdot\, 0 & 1/5 & 0 \\ 0 & 1 & 1 & \cdot\, 1 & 0 & 0 \\ 2 & -3 & -3 & \cdot\, 0 & 0 & 1 \end{vmatrix}$$ Multiplying the first row by 1/5

$$-> \begin{vmatrix} 1 & 1/5 & -1/5 & \cdot\, 0 & 1/5 & 0 \\ 0 & 1 & 1 & \cdot\, 1 & 0 & 0 \\ 0 & -17/5 & 13/5 & \cdot\, 0 & -2/5 & 1 \end{vmatrix}$$ Adding −2 times the first row to the third row

$$-> \begin{vmatrix} 1 & 1/5 & -1/5 & \cdot\, 0 & 1/5 & 0 \\ 0 & 1 & 1 & \cdot\, 1 & 0 & 0 \\ 0 & 0 & 4/5 & \cdot\, 17/5 & -2/5 & 1 \end{vmatrix}$$ Adding 17/5 times the second row to the third row

$$-> \begin{vmatrix} 1 & 1/5 & -1/5 & \cdot\, 0 & 1/5 & 0 \\ 0 & 1 & 1 & \cdot\, 1 & 0 & 0 \\ 0 & 0 & 1 & \cdot\, 17/4 & -2/4 & 5/4 \end{vmatrix}$$ Multiply the third row by 5/4

$$\text{—>}\left|\begin{array}{ccccccc} 1 & 1/5 & -1/5 & \cdot & 0 & 1/5 & 0 \\ 0 & 1 & 0 & \cdot & -13/4 & 2/4 & -5/4 \\ 0 & 0 & 1 & \cdot & 17/4 & -2/4 & 5/4 \end{array}\right|$$

Adding －1 times the third row to the second row

$$\text{—>}\left|\begin{array}{ccccccc} 1 & 1/5 & 0 & \cdot & 17/20 & 1/10 & 1/4 \\ 0 & 1 & 0 & \cdot & -13/4 & 2/4 & -5/4 \\ 0 & 0 & 1 & \cdot & 17/4 & -2/4 & 5/4 \end{array}\right|$$

Adding 1/5 times the tird row to the third row

$$\text{—>}\left|\begin{array}{ccccccc} 1 & 0 & 0 & \cdot & 6/4 & 0 & 2/4 \\ 0 & 1 & 0 & \cdot & -13/4 & 2/4 & -5/4 \\ 0 & 0 & 1 & \cdot & 17/4 & -2/4 & 5/4 \end{array}\right|$$

Adding 11/5 times the second row to the first row

Thus, cleaning up the inverse is

$$A^{-1} = 1/4\left|\begin{array}{rrr} 6 & 0 & 2 \\ -13 & 2 & -5 \\ 17 & -2 & 5 \end{array}\right|$$

12/ Leontief input-output closed model:

Departing from the open model we have worked on earlier, the closed model contains no final demand or consumption by consumers. The consumers, or Households, are treated just as any other sector or industry in an economy that produces output (labor forces). The households demand other sector‘s output as input for producing“labor power” and other sectors need labor service as an input of production. This means that we move the household sector from the final-demand column and place it inside the technically interrelated table, that is, make it one of the endogenous sectors.

Viewing households as a production sector, say sector3, expand the input coefficient matrix A to an (n+1) x (n+1) square matrix.

As an illustration, suppose an economy consists of three sectors, agriculture,Manufacture, and household, which are designed as sector1, 2, and sector3, respectively.

Suppose the input coefficient matrix A is

$$A = \left|\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right| = \left|\begin{array}{ccc} 0.2 & 0.5 & 0.6 \\ 0.5 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{array}\right|$$

The element a_{ij} indicates the dollar value of good i required to produce a dollar‘s worth of good

j. The sum of the elements in the j-th column represents the cost of producing a dollar's worth og good j. Since there is no outside sector, all outputs, including labor power, are used up somewhere in the Input-output production process. Any column sum of A must be equal to 1, that is,

$a_{11} + a_{21} + a_{31} = 1 \quad \text{for} \quad j = 1, 2, 3$

Let X_1 be the total quantity of good i producing and $a_{ij}X_j$ be the portion used as input in producing good j. At market equilibrium, the total output X_i must be equal to the total demand for good i,

$$X_i = a_{1j}X_1 + a_{2j}X_2 + a_{3j}X_3, \quad i = 1, 2, 3.$$

where the terms on the right represent the sum of demands for good i by the other industries. We can write out the system:

$$X_1 = a_{11} X_1 + a_{12} X_2 + a_{13} X_3$$
$$X_2 = a_{21} X_1 + a_{22} X_2 + a_{23} X_3$$
$$X_3 = a_{31} X_1 + a_{32} X_2 + a_{33} X_3$$

or in matrix form, the system is thus

$$\begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix}$$

We can rewrite this as:

Or, $X = AX$

This can be rewritten as:

$$(I - A)\, X = 0.$$

where we now have a homogeneous system. For the solution values (X) not to be zero, then the determinant of $(I-A)$ must be vanish, i.e.

$$|I-A| = 0$$

If true, then the system canbe readily solved. As $|I-A| = 0$, we know

That the homogeneous system $(I-A)\,X = 0$ will have a non-trivial solution X-in fact, it will have an infinite number of non-trivial solutions. However, even though we cannot determine

the absolute levels of X that solves this, we can determine their proportionality.
For this closed model , one (or more) component (s) of final demand is (are) treated endogenously. personal consumption expenditures (sometimes referred to as households). This means to assume that we have a fully self-replacing economy,i.e. an economy which produces at least enough of a commodity as is demanded by other industries as an input.

For instance, in a 3x3 system, it can be easily shown that, from $(I-A)X = 0$, where is a homogeneous system. The linear dependence which guaranteed a vanishing determinant will gurantee that we have vanishinh determinant $|I-A| = 0$.
Thus, for a 3x3 system, from $(I-A)X = 0$, we have:

$$\begin{vmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix} \begin{matrix} = 0 \\ = 0 \\ = 0 \end{matrix}$$

We obtain:

$$\begin{aligned} (I - a_{11})X_1 - a_{12}X_2 \quad - a_{13}X_3 &= 0 \\ -a_{21}X_1 + (1-a_{22})X_2 - a_{23}X_3 &= 0 \\ -a_{31}X_1 - a_{32}X_2 + (1-a_{33})X_3 &= 0 \end{aligned}$$

Thus, as we have seen above, the leontief closed model with respect to households is in fact a homogeneous system of linear equations.
Then,

$$|D| = 0, \quad \text{Cofactor matrix} = \begin{vmatrix} 0.47 & 0.38 & 0.31 \\ 0.47 & 0.38 & 0.31 \\ 0.47 & 0.38 & 0.31 \end{vmatrix}$$

$$\text{Cofactor}^T = \begin{vmatrix} 0.47 & 0.47 & 0.47 \\ 0.38 & 0.38 & 0.38 \\ 0.31 & 0.31 & 0.31 \end{vmatrix}$$

$$\begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix} = k_1 \begin{vmatrix} 0.47 \\ 0.38 \\ 0.31 \end{vmatrix} = k_2 \begin{vmatrix} 0.47 \\ 0.38 \\ 0.31 \end{vmatrix} = k_3 \begin{vmatrix} 0.47 \\ 0.38 \\ 0.31 \end{vmatrix}$$

Let X_1 be 1, then, $\begin{vmatrix} X_2 \\ X_3 \end{vmatrix} = 1/0.47 \begin{vmatrix} 0.38 \\ 0.31 \end{vmatrix} = 1/0.47 \begin{vmatrix} 0.38 \\ 0.31 \end{vmatrix} = \begin{vmatrix} 0.8085106 \\ 0.6595740 \end{vmatrix}$

What about prices?

Let A be the 3x3 matrix of unit input coefficients and p the 3-dimensional price vector as follows;

$$A = \begin{vmatrix} 0.2 & 0.5 & 0.3 \\ 0.5 & 0.3 & 0.2 \\ 0.6 & 0.1 & 0.3 \end{vmatrix}$$

P=AP

$$\begin{vmatrix} P_1 \\ P_2 \\ P_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \begin{vmatrix} P_1 \\ P_2 \\ P_3 \end{vmatrix}$$

This is the no surplus price-cost equality. This can be rewritten as:

$(I - A)\, P = 0$

where we now have a homogeneous system.

$$\begin{vmatrix} 0.8 & -0.5 & -0.3 \\ -0.5 & 0.7 & -0.2 \\ -0.6 & -0.1 & 0.7 \end{vmatrix} \begin{vmatrix} P_1 \\ P_2 \\ P_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$|D| = 0$,

$$\text{Cofactor matrix} = \begin{vmatrix} 0.47 & 0.47 & 0.47 \\ 0.38 & 0.38 & 0.38 \\ 0.31 & 0.31 & 0.31 \end{vmatrix}$$

$$\text{Cofactor}^T = \begin{vmatrix} 0.47 & 0.38 & 0.31 \\ 0.47 & 0.38 & 0.31 \\ 0.47 & 0.38 & 0.31 \end{vmatrix}$$

Let $P_1 = 1$, then, $P_2 = 0.80851$, $P_3 = 0.65957$

Note: on Production function

Economic analysis is frequently couched in terms of the Cobb-Douglas production function $q=AK^aL^b$ ($A > 0$; $0 < a, b < 1$) where q is the quantity of output in physical units, K the quantity of capital, and L the quantity of labor. Here a (the output elasticity of capital) measures the percentage change in q for a 1 percent change in K while L is held constant; b (the output elasiticity of labor) is exactly parallel; and A is an efficiency parameter reflecting the level of technology. A strict Cobb-Douglas function, in which a+b=1, exhibits constant returns to scale. A generalized Cobb-Douglas function, in which a+b≠1, exhibits increasing returns to scale if a+b>1 and decreasing returns to scale if a+b<1.

Production function is said to be homogeneous if when each input factor is multiplied by a positive real constant c, the constant can be completely factored out. If the exponent of the factor is 1, the function is homogeneous of degree 1 as in the case of Cobb-Douglas production function $Q=AK^aL^b$, where a+b=1

Mathematically, a function z =f (x, y) is homogeneous of degree n if for all positive real values of c, f (cx, cy)=k^nf (x, y).

For example, z= 8x + 9y is homogeneous of degree 1 because

F (cx, ck) = 8cx + 9cy = c (8x + 9y)

In order to show that a strict Cobb-Douglas production function

$Q=A^aL^b$, where a+b=1, exhibits constant returns to scale, we multiply each of the inputs by a constant c and factor and get

the following;

$Q (cK, cL) = A (cK)^a (cL)^b = Ac^aK^ac^bL^b$

$= c^{a+b} (AK^aL^b) = c^{a+b} (Q)$.

Let us prove that for a linearly homogeneous Cobb-Douglas production function $Q=AK^a L^b$, a = the output elasticity of capital (ε_{QK}) and b = the output elasticity of labor (ε_{QL}).

From the definition of output elasticity,

$$\varepsilon_{QK} = \frac{\delta Q/\delta K}{Q/L} \quad \text{and} \quad \varepsilon_{QL} = \frac{\delta Q/\delta L}{Q/L}$$

Since a+b= 1, let b= 1－ a and let k= K/L. Then

$Q=AK^aL^{1-a} = A (k/L)^a L =Ak^aL$

To find the marginal functions, we take the partial derivatives of the function with respect to each independent variable.

$\delta Q/\delta K = aAK^{a-1}L^{1-a} =aAK^{a-1}L^{-(a-1)} = aA (K/L)^{a-1} =aAk^{a-1}$

$\delta Q/\delta L = (1-a) AK^aL^{-a} = (1-a) A (K/L)^a = (1-a) Ak^a$

To find the average functions,

$Q/K = Ak^aL /K = Ak^a/k =Ak^{a-1}$

$Q/L = Ak^aL/L =Ak^a$

Then we divide the marginal functions by their respective average functions to obtain ε.

$$\varepsilon_{QK} = \frac{\delta Q/\delta K}{Q/K} = \frac{aAk^{a-1}}{Ak^{a-1}} = a$$

$$\varepsilon_{QL} = \frac{\delta Q/\delta L}{Q/L} = \frac{(1-a)\,Ak^{a}}{Ak^{a}} = 1-a = b$$

The elasticity of substitution ρ measures the percentage change in the least-cost (K/L) input ratio resulting from a small percentage change in the input-output ratio (P_L/P_k). It means that the elasticity of substitution measures the relative change in the K/L ratio brought about a relative change in the price ratio P_L/P_k.

$$\rho = \frac{\dfrac{d\,(K/L)}{K/L}}{\dfrac{d\,(PL/PK)}{P_L/P_K}} = \frac{\dfrac{d\,(K/L)}{d\,(P_L/P_K)}}{\dfrac{K/L}{P_L/P_K}} \qquad (1)$$

where $0 \leq \rho \leq$. If ρ=0, there is no substitutability. This is the case in Leontief production function; the two inputs are complements and must be used together in fixed proportions. If $\rho=\infty$, the two goods are perfect substitutes. A Cobb-Douglas production function has a constant elasticity of substitution equal to 1.

Since a and b are constants in the least-cost input ratio $K/L = aP_L/bP_K$, and P_K and P_L are Independent variables, K/L can be considered a function of P_L/P_K.

Noting that in the second ratio of (1), ρ = the marginal function divided by the average function, first we find the marginal function of $K/L = aP_L/bP_K$.

$$\frac{d\,(K/L)}{d\,(P_L/P_K)} = \frac{a}{b}$$

Then find the average function by diviging both sides of $K/L= aP_L/bP_K$ by P_L/P_K..

$$\frac{K/L}{P_L/P_K} = a/b$$

Substituting in (1)

$$\rho = \frac{\dfrac{d\,(K/L)}{d\,(P_L/P_K)}}{\dfrac{K/L}{P_L/P_K}} = \frac{a/b}{a/b} = 1.$$

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