Inertial Frame Dragging by a Rotating Shell to the Background Universe

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Abstract

Inertial frame dragging by a spherical rotating shell with the density $\Delta\rho$ restricted by the two hypersurfaces is investigated in the background universe, the Mach universe in the Brans-Dicke theory, the Lemaitre universe and the Einstein universe in General Relativity, respectively. The dragging ratio between inertial frame at the origin and the rotating shell diverges to infinity when matter of the universe vanishes in the Mach universe. The Lemaitre universe gives finite inertial frame dragging whether matter of the universe exists or not. The Einstein universe is favorable to Mach's ideas, but it rather belongs to the Brans-Dicke theory. We conclude that General Relativity with an *apriori* gravitational constant G is inconsistent with Mach's principle. The results on the rotating shell derived by Thirring, *et al.*, should be rather regarded to be not Machian. We recognize that Mach's principle functions as a guiding principle to select theories of gravitation or cosmological models of the universe.

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I Introduction

Isaac Newton [1] proposed the bucket Gedanken experiment to support his ideas of absolute space or absolute motion. According to him, it is obvious that a change in the surface of the water in a rotating bucket results from its rotation to absolute space, not to the bucket itself. Hence he declared that we can detect absolute motion to absolute space. However, according to Ernst Mach [2], only a relative motion is detectable and we do not know what happens in the bucket in a case that the wall of the bucket becomes much thicker and more massive (see also Berkeley [3]). Mach insisted that appearances of the Coriolis and the centrifugal forces are due to the relative rotation to the whole universe, and that a local body has inertia only in the presence of other matter. Albert Einstein [4], being impressed by Mach's ideas of inertia, tried to construct his theory of gravitation (General Relativity) in such a way. However, he was only partially successful in this respect. It seems that this fundamental problem (Mach's principle) in cosmology remains controversial even up to the present.

Many authors discussed inertial frame dragging by a rotating shell in the framework of General Relativity. H. Thirring [5], L. Bass and F. A. E. Pirani [6], and H. Okamura *et al.* [7], respectively, showed that "the Coriolis force" ($G\sigma$-term) or "the centrifugal force" ($G^2\sigma^2$-term) appear in the vicinity of a rotating and spherical infinitely thin shell (,with a small constant angular velocity σ) in empty space (Minkowski space) by means of the weak-field approximation of Einstein's field equations. D. R. Brill and J. M. Cohen, or L. Lindblom ([8]-[11]) investigated this rotating shell problem in the Schwarzschild solution. C. Soergel-Fabricius [12] examined inertial effects by a rotating small fraction of the masses $\delta\rho$ at each points of the Einstein universe and showed that the dragging degree is proportional to $\delta\rho/\rho$. H. Hönl and C. Soergel-Fabricius [13] surveyed the distance effect of inertia by a rotating infinitely thin shell in the Einstein universe. These investigations indicate that inertial frame dragging evidently appears in the framework of Einstein's theory, but they still seem to be insufficient to substantiate Mach's ideas.

In the Machian point of view, Lausberg's work [14] is worthy of note in series of these investigations. He discussed what perturbation appears in the metric tensor in the first order of an angular velocity when a spherical shell, with the same density as the remaining part of the universe, rotates uniformly in the Einstein universe and showed that inertial frame dragging at the origin increases as the rotating shell becomes thicker, and complete frame dragging is realized when the shell covers the whole universe. However, the Einstein universe is static and not realistic as a cosmological model at present.

One of next possible extensions is naturally that we apply a similar discussion to the Friedmann universe in General Relativity, but we found that correspondent calculations encountered with difficulties [15]. Then, we turned to the Brans-Dicke theory of gravitation [16], and discovered a new exact cosmological solution [17], [18] for the homogeneous and isotropic universe (see also H. Dehen and O. Obregón [19]). We investigated inertial effects in this expanding and closed cosmological model in the framework of the Brans-Dicke theory of gravitation by means of similar methods, and indicated that complete inertial frame dragging surely appears when the shell covers the whole universe [20] (so, let us call this model the Mach universe).

Complete inertial frame dragging means that inertial properties are completely determined by matter of the whole universe itself. This phenomenon gives a dynamical consistency to general covariance of the field equations as a matter of form. We could say in a sense that Mach's ideas are satisfied in particular cosmological models in the framework of Einstein's theory or the Brans-Dicke theory of gravitation. However, a question still remains. What would happen when matter vanishes in the universe?

In this paper, we will focus this aspect (inertia in empty space). We investigate inertial frame dragging by a spherical and finite thin shell rotating to the background universe, the Mach universe in the Brans-Dicke theory, the Lemaitre Universe and the Einstein universe in General Relativity. We examine a behavior of inertial frame at the origin in the presence of matter and in a case of vanishing matter, and make clear the validity of Mach's principle in each theories of gravitation and a role of the gravitational "constant". The applied perturbation method is basically same as Lausber's [14] or Miyazaki's [20] in a time-varying case, but meaning of its result is essentially fresh and notable.

II Mach Universe in the Brans-Dicke Theory

Let us start from the Mach universe in the Brans-Dicke theory of gravitation. We are interested in an essential behavior of inertial frame dragging in each theories, so it is enough to investigate the simplest type of modified theories of gravitation. The field equations of the Brans-Dicke theory are written in our sign convention as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right) \qquad (1)$$

$$-\frac{1}{\phi}(\phi_{,\,\mu;\,\nu} - g_{\mu\nu}\Box\phi)\,,$$

$$\Box\phi = -\frac{8\pi}{(3+2\omega)c^4}T\,, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, for the perfect fluid,

$$T_{\mu\nu} = -pg_{\mu\nu} - (\rho + p/c^2)u_\mu u_\nu\,, \tag{3}$$

in which ρ, p, and u^μ are the density in comoving coordinates, the pressure, and the four velocity respectively. The symbol $\Box$ denotes the generally-covariant d'Alembertian: $\Box\phi \equiv \phi^{,\mu}{}_{;\mu}$, and the letter ω is the coupling parameter between the scalar field and the contracted energy-momentum tensor T.

For the closed homogeneous and isotropic universe the metric form is given as

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]\,. \tag{4}$$

The Brans-Dicke theory has an exact cosmological solution (the Mach universe) for this metric:

$$\begin{aligned} &a(t) = [-2/(2+\omega)]^{1/2}\, t\,, \\ &2\pi^2 a^3(t)\rho(t) = M\,, \\ &\phi(t) = -[8\pi/(3+2\omega)c^2]\rho(t)t^2 \end{aligned} \tag{5}$$

with $\omega < -2$, neglecting the pressure p, where M means a mass of the universe. This solution satisfies a constraint $a(t)\phi(t) = D\,(const.)$, which is significant in our discussions.

We consider in this universe a shell restricted by the two hypersurface $\chi = \chi_0$ and $\chi = \chi_1$, with $0 < \chi_0 < \chi_1 < \pi$. In the present discussion, only a part of the shell, with the density $\Delta\rho$, rotates slowly around the axis $\theta = 0$ with an angular velocity $\sigma_s = c(d\varphi/dt)_{shell}$ relative to matter of the universe. Remark that the remaining part of the shell with the density $\rho - \Delta\rho$ does not rotate in this universe. The metric form in the whole universe will be perturbed by this partial rotation of the shell as

$$ds^2 = -dt^2 + a^2(t)\left\{d\chi^2 + \sin^2\chi\left[d\theta^2 + \sin^2\theta\,(d\varphi - \sigma dt/c)^2\right]\right\}\,. \tag{6}$$

Owing to the slow rate of rotation ($c^2 \gg a^2(t)\sigma_s^2 > a^2(t)\sigma^2$), it is sufficient to calculate up to the first order of angular velocities σ and σ_s, then we obtain the perturbed metric form as

$$\begin{aligned} ds^2 \;=\; & -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)] \\ & -2\sigma(\chi,\theta,t)a^2(t)\sin^2\chi\sin^2\theta d\varphi dt/c\,. \end{aligned} \tag{7}$$

The nonvanishing components of the energy-momentum tensor are

$$T_{00} = -\rho c^2 \,, \tag{8}$$

$$T_{30} = T_{03} = \begin{cases} -\rho c\sigma a^2 \sin^2\chi \sin^2\theta \quad (\textit{outside of the shell}) \,, \\ -[(\rho-\Delta\rho)c\sigma - \Delta\rho c(\sigma-\sigma_s)]a^2 \sin^2\chi \sin^2\theta \quad (\textit{inside volume}) \,, \end{cases}$$

and the contracted energy-momentum tensor remains $T = \rho c^2$.

After simple calculations of the field equations, we get the following component, which determines inertial effects in the present situation:

$$\begin{aligned} G_{30} &= G_{03} = -(\sigma/c)\sin^2\chi\sin^2\theta(2a\ddot{a}+3) \\ &\quad -(1/2c)\sin^2\chi\sin^2\theta(\partial^2_{\chi\chi}\sigma + 4\cot\chi\partial_\chi\sigma - 4\sigma) \\ &\quad -(1/2c)\sin^2\theta(\partial^2_{\theta\theta}\sigma + 3\cot\theta\partial_\theta\sigma) \\ &= \left[\frac{8\pi}{c^2}(\rho\sigma - \Delta\rho\sigma_s)\frac{a^2}{\phi} + \frac{\omega\sigma}{2c}a^2\left(\frac{\dot\phi}{\phi}\right)^2 - \frac{\sigma}{c}a\dot{a}\left(\frac{\dot\phi}{\phi}\right) + \frac{8\pi\sigma}{(3+2\omega)c^3}\frac{a^2\rho}{\phi}\right] \times \\ &\quad \sin^2\chi\sin^2\theta \quad (\sigma_s = 0,\ \textit{outside the shell}). \end{aligned} \tag{9}$$

By means of other components of the field equations, we can reduce Eq.(9) to

$$\begin{aligned} &\sigma\sin^2\chi\left[\frac{32\pi a^2\rho}{3c^2\phi} + \frac{1}{6}\omega a^2\left(\frac{\dot\phi}{\phi}\right)^2 - 1\right] \\ &-\frac{1}{2}\sin^2\chi(\partial^2_{\chi\chi}\sigma + 4\cot\chi\partial_\chi\sigma) - \frac{1}{2}(\partial^2_{\theta\theta}\sigma + 3\cot\theta\partial_\theta\sigma) \\ = \ &\sigma_s\sin^2\chi\frac{8\pi a^2\Delta\rho}{c^2\phi} \quad (\sigma_s = 0,\ \textit{outside the shell}). \end{aligned} \tag{10}$$

Using relations of the cosmological solution, the inside of the middle bracket of the first term in the left hand side of Eq.(10) becomes

$$\frac{32\pi a^2\rho}{3c^2\phi} + \frac{1}{6}\omega a^2\left(\frac{\dot\phi}{\phi}\right)^2 - 1 = \frac{8\pi a^2\rho}{c^2\phi} = \frac{4M}{\pi c^2 D} = \textit{const.}\,, \tag{11}$$

and thus the homogeneous equation of Eq.(10) admits a variable separation with respect to χ, θ and t, then we can write

$$\sigma(\chi,\theta,t) = X(\chi)\Theta(\theta)/a^2(t)\,. \tag{12}$$

A particular solution of the inhomogeneous equation (10) inside the shell is obviously

$$\sigma_p = (\Delta\rho/\rho)\sigma_s\,. \tag{13}$$

The induced perturbation σ must be regular at $\chi = 0$ and $\chi = \pi$, and so the complete solution $\sigma(\chi, \theta, t)$ is described as

$$\begin{aligned} \sigma_a &= \frac{A}{a^2(t)} X_a(\chi) \quad (0 \leqslant \chi \leqslant \chi_0)\,, \\ \sigma_b &= \frac{1}{a^2(t)} \left[B_1 X_{b_1}(\chi) + B_2 X_{b_2}(\chi)\right] + \sigma_p \quad (\chi_0 \leqslant \chi \leqslant \chi_1)\,, \\ \sigma_a &= \frac{C}{a^2(t)} X_c(\chi) \quad (\chi_1 \leqslant \chi \leqslant \pi)\,, \end{aligned} \tag{14}$$

where

$$\begin{aligned} X_a(\chi) &= F(\alpha, \beta, \alpha + \beta - \gamma + 1; (1 - \cos\chi)/2)\,, \\ X_{b_1}(\chi) &= X_c(\chi) = F(\alpha, \beta, \gamma; (1 + \cos\chi)/2)\,, \\ X_{b_2}(\chi) &= \left(\frac{1 + \cos\chi}{2}\right)^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; (1 + \cos\chi)/2)\,, \end{aligned} \tag{15}$$

$$\alpha + \beta = 4\,, \quad \alpha\beta = \frac{8M}{\pi c^2 D}\,, \quad \gamma = \frac{5}{2}\,, \tag{16}$$

and A, B_1, B_2, and C are arbitrary constants, which are determined by means of the conditions that σ_a, σ_b, and σ_c must be connected smoothly at $\chi = \chi_0$ and $\chi = \chi_1$. The expression $F(\alpha, \beta, \gamma; x)$ denotes Gauss' hypergeometric function. In order that the equation (14) is consistent at all time, the particular solution σ_p must have the form as $\Omega/a^2(t)$ ($\Omega = const.$). We suppose that the density $\Delta\rho$ of the rotating shell also obeys the expansion of the universe and that the total mass ΔM of the shell remains constant, satisfying the conservation of the angular momentum.

We are interested in the metric form in the vicinity of the origin, so it is enough to determine only a value of A. The solution inside the shell ($0 \leqslant \chi \leqslant \chi_0$) becomes

$$\sigma_a(\chi_0, \chi_1; \chi, t) = \frac{Q(\chi_0, \chi_1)}{P(\chi_0, \chi_1) a^2(t)} F(\alpha, \beta, \alpha + \beta - \gamma + 1; (1 - \cos\chi)/2)\,, \tag{17}$$

where

$$P(\chi_0, \chi_1) = \begin{vmatrix} X_a(\chi_0) & -X_{b_1}(\chi_0) & -X_{b_2}(\chi_0) & 0 \\ 0 & -X_{b_1}(\chi_1) & -X_{b_2}(\chi_1) & X_c(\chi_1) \\ X'_a(\chi_0) & -X'_{b_1}(\chi_0) & -X'_{b_2}(\chi_0) & 0 \\ 0 & -X'_{b_1}(\chi_1) & -X'_{b_2}(\chi_1) & X'_c(\chi_1) \end{vmatrix}\,, \tag{18}$$

$$Q(\chi_0,\chi_1) \;=\; \begin{vmatrix} \Omega & -X_{b_1}(\chi_0) & -X_{b_2}(\chi_0) & 0 \\ \Omega & -X_{b_1}(\chi_1) & -X_{b_2}(\chi_1) & X_c(\chi_1) \\ 0 & -X'_{b_1}(\chi_0) & -X'_{b_2}(\chi_0) & 0 \\ 0 & -X'_{b_1}(\chi_1) & -X'_{b_2}(\chi_1) & X'_c(\chi_1) \end{vmatrix} ,$$

and a prime denotes ∂_χ. The function $Q(\chi_0,\chi_1)$ is proportional to Ω, so we can rewrite $Q(\chi_0,\chi_1)/P(\chi_0,\chi_1) = R(\chi_0,\chi_1)\Omega$, which gives a distance effect of the shell. Then, at the origin, we get

$$\sigma_0(\chi_0,\chi_1;t) = \lim_{\chi\to 0}\sigma_a(\chi_0,\chi_1;\chi,t) = R(\chi_0,\chi_1)\sigma_p. \tag{19}$$

This gives the induced angular velocity of inertial frame at the origin by the rotating shell with the density $\Delta\rho$ and the angular velocity σ_s restricted by the two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$. Thus, we obtain our final result

$$\sigma_0/\sigma_s = R(\chi_0,\chi_1)\Delta\rho/\rho\,, \tag{20}$$

or

$$\sigma_0/\sigma_s = (2\pi a^2/c^2)R(\chi_0,\chi_1)G(\rho)\Delta\rho\,, \tag{21}$$

where the gravitational "constant" $G = [(4+2\omega)/(3+2\omega)]\phi^{-1}$.

We find that the value of $R(\chi_0,\chi_1)$ converges to 1 when the shell covers the whole universe ($\chi_0\to 0$ and $\chi_1\to\pi$), by means of asymptotic behaviors of the hypergeometric function at $\chi=0$ and $\chi=\pi$, and confirm surely that the dragging ratio of inertial frame between the induced angular velocity σ_0 at the origin and σ_s of the rotating shell becomes unity if the density $\Delta\rho$ of the shell coincides with the density ρ of the universe. The most significant behavior to be stressed here is that the dragging degree diverges to infinity when the density of the universe converges to 0 (at this time, the gravitational "constant " $G(\rho)$ simultaneously diverges to infinity). This means that the Mach universe in the Brans-Dicke theory loses its inertial property when matter vanishes in the universe, and seems to be the most affirmative evidence for Mach's principle.

III Lemaitre Universe in General Relativity

We extensively discuss here the Lemaitre universe, rather than the Friedmann universe. The Einstein theory of gravitation with the cosmological constant λ is described by the field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \lambda g_{\mu\nu} = \kappa T_{\mu\nu}\,, \tag{22}$$

where κ is Einstein's gravitational constant and for the closed homogeneous and isotropic universe it has the well-known solution

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{23}$$

$$\begin{aligned} &2\pi^2 a^3(t)\rho(t) = M\,, \\ &\dot{a}^2 = \kappa M c^2/(6\pi^2 a) + (1/3)\lambda a^2 - 1\,, \end{aligned} \tag{24}$$

which determines time-variations of the expansion parameter and the density of the universe (the Friedmann equation).

We consider the same rotating shell with the density $\Delta\rho$ in this universe, and then obtain, through similar procedures, the perturbed component of the field equations

$$\begin{aligned} G_{30} &= G_{03} = -(\sigma/c)\sin^2\chi\sin^2\theta(2a\ddot{a} - \lambda a^2 + 3) \\ &\quad -(1/2c)\sin^2\chi\sin^2\theta(\partial^2_{\chi\chi}\sigma + 4\cot\chi\partial_\chi\sigma - 4\sigma) \\ &\quad -(1/2c)\sin^2\theta(\partial^2_{\theta\theta}\sigma + 3\cot\theta\partial_\theta\sigma) \\ &= -\kappa c(\rho\sigma - \Delta\rho\sigma_s)a^2\sin^2\chi\sin^2\theta \quad (\sigma_s = 0,\ \textit{outside the shell}), \end{aligned} \tag{25}$$

which is reduced, by means of other components of the field equations and relations of the cosmological solution, to

$$\begin{aligned} &\sigma\sin^2\chi(\kappa a^2\rho c^2 + \dot{a}^2) \\ &-\frac{1}{2}\sin^2\chi(\partial^2_{\chi\chi}\sigma + 4\cot\chi\partial_\chi\sigma) - \frac{1}{2}(\partial^2_{\theta\theta}\sigma + 3\cot\theta\partial_\theta\sigma) \\ = \; &\sigma_s\sin^2\chi\kappa a^2\Delta\rho c^2 \quad (\sigma_s = 0,\ \textit{outside the shell}). \end{aligned} \tag{26}$$

Though some problems appear in analysis of this equation, we already know that it is for the particular solution of this inhomogeneous equation to play important roles, and get

$$\sigma_p = \frac{\kappa a^2\Delta\rho c^2}{\kappa a^2\rho c^2 + \dot{a}^2}\sigma_s\,. \tag{27}$$

We pay attention to an asymptotic behavior of inertial frame dragging when matter of the universe vanishes. Let the density ρ of the universe converge to 0, and so we obtain roughly as the dragging ratio

$$\sigma_0/\sigma_s \approx R(\chi_0, \chi_1)(\dot{a}/a)^{-2}\kappa\Delta\rho c^2\,, \tag{28}$$

where $\dot{a}/a$ means the Hubble constant. Therefore, finally, we get the relation

$$\sigma_0/\sigma_s \propto G\Delta\rho\,, \tag{29}$$

which is not Machian. Because this relation indicates that empty space has inertial properties, that is, the *apriori* gravitational constant G gives inertial frame dragging even if matter of the universe vanishes. In the Friedmann or Lemaitre universe, the gravitational constant is given first in the theory and the density of the universe is arbitrary (, in its range of the phase). We suppose that the equation (28) or Eq. (29) are not modified, on the whole, by its phase of the universe. Anyway, the equation (29) indicates that inertial frame dragging appears in General Relativity whether matter of the universe exists or not.

When the expansion parameter satisfies $\dot{a} = 0\,(\rho \neq 0)$, the Lemaitre universe, with the cosmological constant, behaves like the Einstein universe, and then we get the relation from Eq.(27)

$$\sigma_0/\sigma_s \approx R(\chi_0, \chi_1)\Delta\rho/\rho\,. \tag{30}$$

If we converges $\rho \to 0$ in this static state ($\lambda a^2 \to 3$), we find the dragging degree to diverge to infinity. It seems that the static universe has an exceptional position in General Relativity, because of the *apriori* gravitational constant.

IV Einstein Universe in General Relativity

The static Einstein universe is, when the pressure of the universe is negligible, given as

$$\begin{aligned} \lambda a^2 &= 1\,, \\ \kappa a^2 \rho c^2 &= 2\,, \end{aligned} \tag{31}$$

for the closed metric form

$$ds^2 = -dt^2 + a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]\,. \tag{32}$$

The equation we should solve concerning inertial frame dragging becomes

$$\begin{aligned} & \sigma\sin^2\chi(\kappa a^2\rho c^2 + \lambda a^2 - 1) \\ & -\frac{1}{2}\sin^2\chi(\partial^2_{\chi\chi}\sigma + 4\cot\chi\partial_\chi\sigma) - \frac{1}{2}(\partial^2_{\theta\theta}\sigma + 3\cot\theta\partial_\theta\sigma) \\ = \;& \sigma_s\sin^2\chi\kappa a^2\Delta\rho c^2 \quad (\sigma_s = 0,\ \textit{outside the shell}), \end{aligned} \tag{33}$$

and we directly find the particular solution of the inhomogeneous equation, using the cosmological relation,

$$\sigma_p = (\Delta\rho/\rho)\sigma_s\,. \tag{34}$$

Thus we get the dragging ratio between the induced inertial frame at the origin and the angular velocity of the rotating shell at a distance

$$\sigma_0/\sigma_s = R(\chi_0, \chi_1)\Delta\rho/\rho\,, \tag{35}$$

or

$$\sigma_0/\sigma_s = (1/2)R(\chi_0, \chi_1)\kappa a^2 c^2 \Delta\rho\,. \tag{36}$$

The equation (35) surely indicates that complete inertial frame dragging appears when the rotating shell covers the whole universe and the density $\Delta\rho$ becomes ρ, and moreover that the dragging degree diverges to infinity when the density of the universe converges to 0. Though these aspects are very affirmative to Mach's ideas, on the other hand, the equation (36) means the expression $\sigma_0/\sigma_s \propto G\Delta\rho$, which rather seems to be not Machian, like Eq.(29) in the Lemaitre universe. It is sure that the expansion parameter a diverges to infinity due to the original relation (31b) of the Einstein universe when the density ρ goes to 0, and the dragging degree may also diverge to infinity. However, this insistence does not seem to make sense. Because the cosmological constant is essential in the Einstein universe. This gives a definite radius a of the universe by Eq.(31a) and thus the given gravitational constant also determines a definite density ρ of the universe through Eq.(31b). The static Einstein universe constitutes the rigid cosmological model. Therefore, the above collision does not disappear, because of the *apriori* gravitational constant in General Relativity. In the first place, it may be more appropriate saying that we can not vary the density ρ of the Einstein universe.

We may also consider, as a possible interpretation, that the gravitational constant should be originally supported by matter of the universe like the Brans-Dicke theory and that the density ρ of the universe produces the present gravitational constant G by Eq.(31b). Thus, when we vary the density ρ, the gravitational "constant" $G(\rho)$ could also behave as a variable, according to $G(\rho) \propto \rho^{-1}$. In the result, the equation (35) and Eq.(36) do not contradict with each other in the Machian point of view, like the Mach universe in the Brans-Dicke theory. Though the static Einstein universe may certainly be regarded as Machian, it is not a cosmological model in General Relativity, but in extended theories of gravitation with the varying gravitational "constant" like the Brans-Dicke theory. In fact the Mach universe in the Brans-Dicke theory corresponds to the static Einstein universe in the limit of the coupling parameter $\omega \to -\infty$ [21].

V Discussions and Concluding Remarks

Since investigations concerning the rotating shell by Thirring, *et al.*, their works have been regarded to be favorable to Mach's ideas; Einstein's theory of gravitation, though partially, realizes his ideas of inertia. However, our result indicates that these understandings are no longer adequate. Their discussions set Minkowski space as the background universe. Nevertheless, finite inertial frame dragging, which is proportional to $G\sigma_s$, is induced at the center of the rotating shell. This aspect means that empty space with no matter has inertial properties (: original inertial frame is fixed to the empty background universe), and it is nothing but the vestiges of Newton's absolute space. Therefore these results should be rather regarded to be not Machian. On the contrary of our expectations, General Relativity with an *apriori* gravitational constant G fundamentally contradicts Mach's principle.

However General Relativity never lose its validity in general problems concerning space and time. General Relativity remains to be the extremely beautiful and precise theory of gravitation, being supported by many observational and experimental results. After all, Mach's principle is completely a *cosmological* problem and requires the presence of matter in the universe for physical consistency. The presence of matter in the universe enables us to detect motion to each other and influences inertial effects, which is closely related to the gravitational constant. General Relativity is a theory of gravitation with an *apriori* gravitational constant, which is *actually* supported by matter of the present universe. Therefore, in fact rotating shell problems by Thirring, *et al.*, are not treated in empty space, but in prerequisite matter of the universe, and they could partially induce inertial frame dragging in the framework of General Relativity. Einstein's theory of gravitation can not originally handle Mach's ideas because of its prerequisite.

We judge that Mach's principle functions as a guiding principle to select theories of gravitation or cosmological models of the universe. We can regard that it excludes General Relativity from a proper fundamental theory of gravitation and so that the Friedmann universe, the Lemaitre universe, and even the Einstein universe, in General Relativity, are actually not Machian as cosmological models. It seems that the Brans-Dicke theory is the first prototype of gravitational theories to conform to Mach's ideas. The Mach universe (, we call) is to be expected as an exact closed cosmological model to accord with them. In this model, complete inertial frame dragging when a spherical rotating shell covers the whole universe is realized and moreover the dragging ratio between inertial frame at the origin and the spherical shell, rotating relatively to the remaining part of the universe, diverges to infinity when matter of the universe vanishes. This second fact, which might be more essential than the first one, is precisely that we sought in this paper. We realized that the static Einstein universe actually belongs to the cosmological models in the Brans-Dicke theory.

In these discussions, the varying gravitational "constant" plays an important function. It is especially most significant that the gravitational constant diverges to infinity when matter of the universe vanishes. We have, in general, regarded that the Brans-Dicke theory should go to Einstein's theory when the coupling parameter ω of the scalar field diverges to infinity. However, in the Machian point of view, we should abandon this fixed idea. This correspondence is derived from the *apriori* scalar field free of the coupling parameter or matter of the universe and we know that cosmological models with the *apriori* gravitational constant have inertial properties even in empty space. Cosmological models which have correspondences in General Relativity are considered to be not Machian. Concerning the varying gravitational "constant", in another interpretation, we can make it be included to inertial mass of particles through a conformal transformation of the metric [22].

It seems that the Mach universe, as a prototype, has more than inertial properties discussed here. The remarkable aspect of this model is that it satisfies, although the universe expands forever, the relation $G(t)M/c^2a(t) = \pi$, which is closely connected with the asymptotic behavior of the scalar field $\phi = O(\rho/\omega)$ for the large coupling parameter. Furthermore, we extended this cosmological model for the generalized scalar-tensor theory of gravitation with a variable coupling parameter and a cosmological constant, and discussed a whole scenario of our universe in the framework of this theory [23]. However, even now it remains as a physical defect that the coupling function must be *negative* to derive the attractive gravitational force for the closed cosmological model. Probably a true scalar-tensor theory of gravitation, in classical meaning, would be consistently derived from an ultimate theory, like a superstring theory. Quantum fields induced there or vacuum fluctuations also should be included to the density ρ of the universe. Even in this situation, Mach's principle would play an important role to select a physical model of the universe.

In this paper, we considered a spherical rotating shell with the density $\Delta\rho$ restricted by the two hypersurfaces, and so the remaining part of the shell with the density $\rho - \Delta\rho$ remains to be static to universe. This setting enabled us to investigate only a perturbation of (3,0)-component of the field equations in the same background universe with the density ρ. However, strictly speaking, it is not appropriate that we handle the limit $\rho \to 0$ concerning a behavior of inertial frame dragging in this situation. We should rather set the additional rotating shell with the density $\Delta\rho$ in the considering universe. If so, we need consider perturbations to the structure itself of the universe by the rotating shell, and these aspects would complicate the necessary calculations. We unavoidably regard that the density $\Delta\rho$ of the rotating shell is so small that we can neglect its extra perturbations enough, and then discuss the asymptotic behavior of inertial frame dragging when the density of the universe goes to small enough, holding the same structure of the universe. It seems to be natural that a tendency of its behavior continuously goes on to the limit $\rho \to 0$. We have recognized that inertial frame dragging is in general proportional to $G\sigma_s$ and know that the gravitational "constant " $G(\rho)$ exactly diverges to infinity when $\rho \to 0$, according to cosmological relations in the Mach universe. This fact supports our results derived from the above perturbation.

Inertial frame is defined as system of coordinates in which inertial force does not appear. So when inertial frame dragging is induced by a rotating shell, inertial force appears in its original coordinate system and acts on particles to accelerate. Then, what is its counteraction? What does its counteraction act upon? Our answer is matter of the universe itself. Counteraction of inertial force acts upon matter of the universe, and matter of the universe inversely induces inertial force in coordinate system accelerating to "*matter of the universe*". Thus the background universe dominates inertial frame in the presence of matter.

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