On the quantities W, L, K derived from linear parallel displacements in Finsler geometry

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Abstract

Last year the author studied linear parallel displacements along an infinitesimal parallelogram and obtained objects are regarded as evaluating "a curvature" at each point on a Finsler space([8]). In this paper, their properties are investigated in detail.

Keywords and phrases : linear parallel displacement, infinitesimal parallelogram, locally Minkowski space, Finsler geometry.

Introduction

The author have been studying linear parallel displacements in Finsler geometry from 2008. Last year the author studied objects provided from two linear parallel displacements. One is a difference of two parallel vectors provided from linear parallel displacements along two one-way courses of an infinitesimal parallelogram. The other is a difference between an initial vector and a vector provided from a linear parallel displacement going around an infinitesimal parallelogram. Then three quantities are found out from the objects. The author call them "W, L, K" in §2. These W, L, K are Finsler tensor fields whose coefficients are functions on the subbundle $\mathcal{H}$ of TTM. In §3, locally Minkowski space is distinguished by W, L, K(Theorem 3.2).

Here, we put terminology and notations used in this paper(cf.[1] and [2]). Let M be an n-dimensional differentiable manifold and $x = (x^i)$ a local coordinate of M. TM is the tangent bundle of M and $(x, y) = (x^i, y^i)$ is a local coordinate of TM. $N = (N^i_j(x, y))$ is an non-linear connection of TM and its coefficients of N on a local coordinate (x, y). $F(x, y)$ is a Finsler structure (or Finsler metric, Finsler fundamental function) on M. Further, $F\Gamma = (N^i_j(x, y), F^i_{jr}(x, y), C^i_{jr}(x, y))$ is Finsler connection and its coefficients of $F\Gamma$ satisfying $T^i_{rj} := F^i_{rj} - F^r_{jr} = 0$, $D^i_j := y^r F^i_{rj} - N^i_j = 0$ and $g_{ij|k}(x, y) = 0$(h-metrical). And $N^i_j(x, y), F^i_{jk}(x, y), C^i_{rj}(x, y)$ are positively homogeneous of degree 1, 0 and -1, respectively. Therefore N^i_j and F^i_{jr} come to Cartan's ones. Last, we denote the collection of horizontal vectors at every point on TM by $\mathcal{H}$. This is the subbundle of TTM and its dimension is $3n$. So we denote a local coordinate of $\mathcal{H}$ by (x, y, z). And it is called "horizontal subbundle of TTM". All of objects appeared in this paper (curves, vector fields, etc) are differentiable. In additions, indexes $a, b, c, \cdots, h, i, j, k, l, m, \cdots, \alpha, \beta, \cdots$, run on from 1 to $n = \dim M$.

1 Linear parallel displacement

Now, for a vector field on a curve c with a parameter t, we give a following definition of linear parallel displacements along c([3],[4],[5],[6]).

Definition 1.1 *For a curve $c = (c^i(t))$ $(a \leq t \leq b)$ on M and a vector field $v = (v^i(t))$ along c, if the equation*

$$(1.1) \qquad \frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c})\dot{c}^r = 0 \qquad (\dot{c}^r = \frac{dc^r}{dt})$$

is satisfied, then v is called a parallel vector field along c, and we call the linear map $\Pi_c : \; v(a) \longrightarrow v(b)$ a linear parallel displacement along c.

The difference from the traditional notion of parallel displacements in Finsler geometry are three points. One of them is the fact that the inverse vector field $v^{-1}(\tau)(\tau = -t+a+b)$ is not always parallel along the inverse curve $c^{-1}(\tau)$, even if $v(t)$ is parallel vector field along a curve $c(t)$(cf.[3], [5]). For this point, we have a following theorem.

Theorem 1.1 *Let $c(t)$ be a curve on M and $v(t)$ parallel vector field along c. The inverse vector field $v^{-1}(\tau)(\tau = -t+a+b)$ is always parallel along the inverse curve $c^{-1}(\tau)$, if and only if*

$$(1.2) \qquad F^i_{0j}(c, \dot{c}) + F^i_{0j}(c, -\dot{c}) = 0$$

is satisfied.

The others of them are facts that we can consider, for vector fields $u(t)$, $v(t)$ along $c(t)$, an inner product $g_{ij}(c, \dot{c})u^i(t)v^j(t)$ along the curve $c(t)$ and the inner product is not always preserved, even if u, v are parallel vector fields along c.

Then we have(cf.[4])

Proposition 1.1 *Let $(M, F(x, y))$ be a Finsler space with a Finsler connection $(N^i_j, F^i_{jr}, C^i_{jr})$ satisfying h-metrical $g_{ij|r} = 0$. For any parallel vector field $v = (v^i(t))$, $u = (u^i(t))$ along a curve $c = (c^i(t))$, if c is a path or a geodesic, then the inner product $g_{ij}(c, \dot{c})v^i u^j$ along c is preserved.*

Theorem 1.2 *Let $(M, F(x, y))$ be a Finsler space with a Finsler connection. We assume that the Finsler connection is h-metrical and the metric g_{ij} is positive definite. Any smooth curve preserves the inner products of parallel vector fields along it, if and only if $\frac{\partial g_{ij}}{\partial y^r} = 0$ is satisfied, namely, (M, g_{ij}) is a Riemannian space.*

2 Tensor fields W, L, K

Next we introduce conclusions obtained by investigating linear parallel displacements along an infinitesimal parallelogram([8]).

We studied two cases. One is the case that makes an initial vector be two parallel vector fields along two routes(Case I), and the other is the case making a parallel vector field along one loop(Case II). Hereafter, we assume that all points and curves are in one coordinate neighborhood.

Case I. Let p, q, r, s be four points on M and $(x^i), (x^i + \xi^i), (x^i + \xi^i + \eta^i), (x^i + \eta^i)$ coordinates, respectively. Further c_1, c_2, c_3 and c_4 are following curves with a parameter t $(0 \leq t \leq 1)$:

$$(I) \begin{cases} c_1(t): & x^i(t) = x^i + t\xi^i \ (p \text{ to } q), \\ c_2(t): & x^i(t) = x^i + \xi^i + t\eta^i \ (q \text{ to } r), \\ c_3(t): & x^i(t) = x^i + t\eta^i \ (p \text{ to } s), \\ c_4(t): & x^i(t) = x^i + \eta^i + t\xi^i \ (s \text{ to } r). \end{cases}$$

We take two routes $c = c_1 + c_2 (p \to q \to r)$ and $\bar{c} = c_3 + c_4 (p \to s \to r)$, and consider linear parallel displacements along c and $\bar{c}$, respectively. Let $V = (V^i)$ be an initial vector at p and V_q, V_r the value at q and r by the parallel vector field along c, respectively. Further, let $\bar{V}_s, \bar{V}_r$ the value at s and r by the parallel vector field along $\bar{c}$, respectively(See Figure 1).

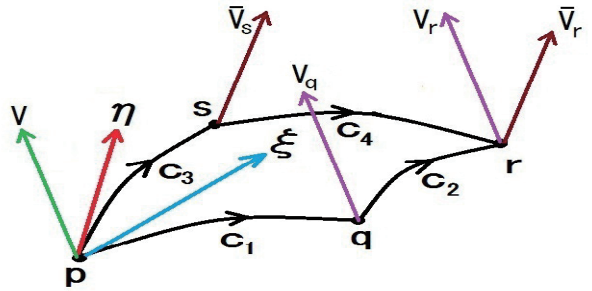

Figure 1: Case I

Our standpoint is to investigate the difference $\bar{V}_r - V_r$. The result is as follows:

(2.1)

$$\bar{V}^i_r - V^i_r = [(F^i_{hj}(x, \xi) - F^i_{hj}(x, \eta))(\xi^j + \eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x, \xi)\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x, \eta)\xi^j)(\eta^k - \xi^k)$$
$$+ (\frac{\delta F^i_{hj}}{\delta x^k}(x, \xi) - \frac{\delta F^i_{hk}}{\delta x^j}(x, \eta) - F^i_{mj}(x, \xi)F^m_{hk}(x, \xi) + F^i_{mk}(x, \eta)F^m_{hj}(x, \eta))\eta^j\xi^k]V^h + \cdots,$$

Case II. Let four points p, q, r, s be the same in Case I. The curves c_3, c_4, however, are different from (I) as follows

$$(II) \begin{cases} c_1(t): & x^i(t) = x^i + t\xi^i \ (p \text{ to } q), \\ c_2(t): & x^i(t) = x^i + \xi^i + t\eta^i \ (q \text{ to } r), \\ c_3(t): & x^i(t) = x^i + \xi^i + \eta^i - t\xi^i \ (r \text{ to } s), \\ c_4(t): & x^i(t) = x^i + \eta - t\eta^i \ (s \text{ to } p), \end{cases}$$

where $0 \leq t \leq 1$.

We take a loop $c = c_1 + c_2 + c_3 + c_4 (p \to q \to r \to s \to p)$ and consider a linear parallel displacement along c. Let $V = (V^i)$ be an initial vector at p and V_q, V_r, V_s the value of the parallel vector field along c at q, r, s, respectively. Further, let $\bar{V}$ be the value at the end point p(See Figure 2).

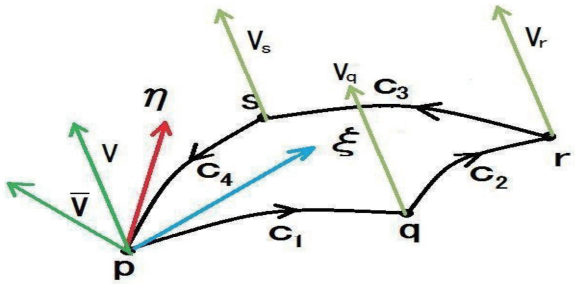

Figure 2: Case II

Our standpoint is to investigate the difference $\bar{V} - V$. The result is as follows:

(2.2)

$$\bar{V}^i - V^i = [(F^i_{hj}(x,-\eta) - F^i_{hj}(x,\xi))(\xi^j + \eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(-\eta^j))(\eta^k - \xi^k)$$
$$+ (\frac{\delta F^i_{hj}}{\delta x^k}(x,-\eta) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\xi) - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta) + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi))\xi^j\eta^k]V^h + \cdots .$$

Remark 2.1 *In* (2.1) *and* (2.2), $(\cdots)$ *expresses 3rd and more degree terms with respect to* ξ, η.

In Riemannian geometry, two differences $\bar{V}^i_r - V^i_r$ and $\bar{V}^i - V^i$ are evaluated by "single" Riemannian curvature at the point. But in our Finsler case, we notice that the quantity is not as same as that in Riemannian case from (2.1) and (2.2). So we can consider following quantities by (2.1)

$$W^i_{hj}(x,\xi,\eta) := F^i_{hj}(x,\xi) - F^i_{hj}(x,\eta), \tag{2.3}$$

$$L^i_{hj}(x,\xi,\eta) := \frac{\partial F^i_{hk}}{\partial y^j}(x,\xi)\eta^k + \frac{\partial F^i_{hk}}{\partial y^j}(x,\eta)\xi^k, \tag{2.4}$$

$$K^i_{hjk}(x,\xi,\eta) := \frac{\delta F^i_{hj}}{\delta x^k}(x,\xi) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\eta) - F^i_{mj}(x,\xi)F^m_{hk}(x,\xi) + F^i_{mk}(x,\eta)F^m_{hj}(x,\eta). \tag{2.5}$$

Then we can express (2.1) as follows:

$$\bar{V}^i_r - V^i_r = [W^i_{hj}(x,\xi,\eta)(\xi^j + \eta^j) + L^i_{hj}(x,\xi,\eta)(\eta^j - \xi^j) + K^i_{hjk}(x,\xi,\eta)\eta^j\xi^k]V^h + \cdots . \tag{2.6}$$

On the other hand, we can consider following quantities by (2.2)

$$W^i_{hj}(x,-\eta,\xi) := F^i_{hj}(x,-\eta) - F^i_{hj}(x,\xi), \tag{2.7}$$

$$L^i_{hj}(x,-\eta,\xi) := \frac{\partial F^i_{hk}}{\partial y^j}(x,-\eta)\xi^k + \frac{\partial F^i_{hk}}{\partial y^j}(x,\xi)(-\eta^k), \tag{2.8}$$

(2.9)

$$K^i_{hjk}(x,-\eta,\xi) := \frac{\delta F^i_{hj}}{\delta x^k}(x,-\eta) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\xi) - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta) + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi).$$

Then we also can express (2.2) as follows:

(2.10)

$$\bar{V}^i - V^i = [W^i_{hj}(x,-\eta,\xi)(\xi^j+\eta^j) + L^i_{hj}(x,-\eta,\xi)(\eta^j-\xi^j) + K^i_{hjk}(x,-\eta,\xi)\xi^j\eta^k]V^h + \cdots .$$

After all, we have a following theorem([8]).

Theorem 2.1 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk}(x,y)=0,\ D^i_j(x,y)=0$. First, for an infinitesimal parallelogram defined by (I) and an initial vector $V=(V^i)$, we have the difference $\bar{V}_r - V_r$ satisfies* (2.6)*. Next, for an infinitesimal parallelogram defined by (II) and an initial vector $V=(V^i)$, the parallel vector $\bar{V}=(\bar{V}^i)$ is obtained and the differences $\bar{V}-V$ satisfies* (2.10).

Remark 2.2 *We know a curvature transformation in Riemannian geometry. In our case, following quantities*

$$W^i_{hj}(x,\xi,\eta)(\xi^j+\eta^j) + L^i_{hj}(x,\xi,\eta)(\eta^j-\xi^j) + K^i_{hjk}(x,\xi,\eta)\eta^j\xi^k \tag{2.11}$$

$$W^i_{hj}(x,-\eta,\xi)(\xi^j+\eta^j) + L^i_{hj}(x,-\eta,\xi)(\eta^j-\xi^j) + K^i_{hjk}(x,-\eta,\xi)\xi^j\eta^k \tag{2.12}$$

may be regarded as it. we, however, can't decide which is "curvature transformation" in Finsler geometry, yet.

Now we investigate $W^i_{hj}(x,-\eta,\xi), L^i_{hj}(x,-\eta,\xi), K^i_{hjk}(x,-\eta,\xi)$. These quantities are obtained from the case making a parallel vector field along one loop(Case II) but are also given from the case that makes an initial vector be two parallel vector fields along two routes defined by $(-\eta,\xi)$, likely Case I(See Figure 3).

Further, we can consider cases of $(-\xi,-\eta)$ and $(\eta,-\xi)$. Then we have

$$W^i_{hj}(x,-\xi,-\eta),\ L^i_{hj}(x,-\xi,-\eta),\ K^i_{hjk}(x,-\xi,-\eta), \tag{2.13}$$

$$W^i_{hj}(x,\eta,-\xi),\ L^i_{hj}(x,\eta,-\xi),\ K^i_{hjk}(x,\eta,-\xi). \tag{2.14}$$

Therefore after all, we can define following three quantities

$$W^i_{hj}(x,y,z) := F^i_{hj}(x,y) - F^i_{hj}(x,z), \tag{2.15}$$

$$L^i_{hj}(x,y,z) := \frac{\partial F^i_{hm}}{\partial y^j}(x,y)z^m + \frac{\partial F^i_{hm}}{\partial z^j}(x,z)y^m, \tag{2.16}$$

(2.17)

$$K^i_{hjk}(x,y,z) := \frac{\delta F^i_{hj}}{\delta x^k}(x,y) - \frac{\delta F^i_{hk}}{\delta x^j}(x,z) - F^i_{mj}(x,y)F^m_{hk}(x,y) + F^i_{mk}(x,z)F^m_{hj}(x,z).$$

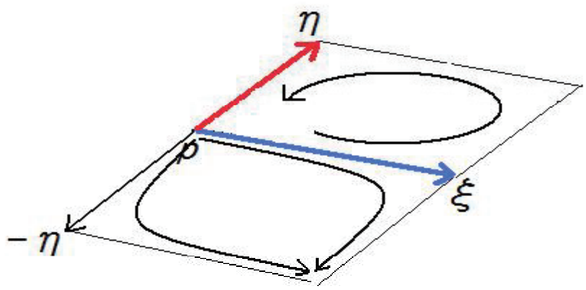

Figure 3: (ξ, η) and $(-\eta, \xi)$

Then the formulas (2.3) $\sim$ (2.5), (2.7) $\sim$ (2.9), (2.13) and (2.14) are regarded as the values at four points (x, ξ, η), $(x, -\eta, \xi)$, $(x, -\xi, -\eta)$, $(x, \eta, -\xi)$ on $\mathcal{H}$, respectively(See Figure 4).

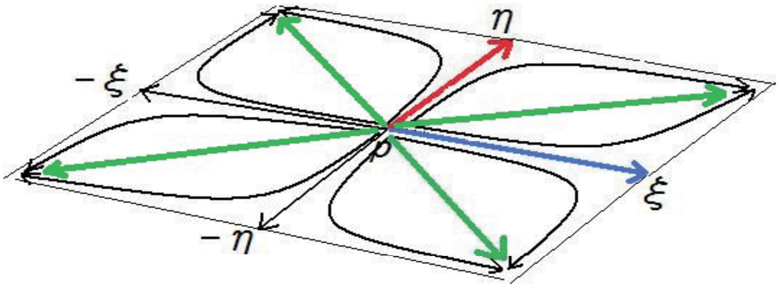

Figure 4: Values at (x, ξ, η), $(x, -\eta, \xi)$, $(x, -\xi, -\eta)$, $(x, \eta, -\xi)$

3 Properties of W, L, K

Now we investigate properties of W, L, K in detail. First the tensorial property of W, L, K are trivial by the definitions (2.15), (2.16) and (2.17) because of the positively homogeneity and the formation of coordinate transformations of $F^i_{hj}(x, y)$.

Since the positively homogeneous of degree 0 and -1 of $F^i_{hr}(x, y)$ and $\frac{\partial F^i_{hm}}{\partial y^j}(x, \xi)$ with respect to y, respectively, we have a following lemma.

Lemma 3.1 *For any positive number* $\rho > 0, \sigma > 0$,

(3.1) $$W^i_{hj}(x, \rho\xi, \sigma\eta) = W^i_{hj}(x, \xi, \eta),$$

(3.2) $$L^i_{hj}(x, \rho\xi, \sigma\eta) = \frac{\sigma}{\rho}\frac{\partial F^i_{hm}}{\partial y^j}(x, \xi)\eta^m + \frac{\rho}{\sigma}\frac{\partial F^i_{hm}}{\partial y^j}(x, \eta)\xi^m,$$

(3.3) $$K^i_{hjk}(x, \rho\xi, \sigma\eta) = K^i_{hjk}(x, \xi, \eta)$$

are satisfied.

Here we notice that if $W^i_{hj}(x,\xi,\eta) = 0,\ L^i_{hj}(x,\xi,\eta) = 0,\ K^i_{hjk}(x,\xi,\eta) = 0$ are satisfied, then $W^i_{hj}(x,\rho\xi,\sigma\eta) = 0,\ K^i_{hjk}(x,\rho\xi,\sigma\eta) = 0$ are true from Lemma 2.1. For L, $L^i_{hj}(x,\rho\xi,\sigma\eta)(\rho\xi^j - \sigma\eta^j) = 0$ is satisfied. Because of:
From $L^i_{hj}(x,\xi,\eta) = 0$, $\frac{\partial F^i_{hm}}{\partial y^j}(x,\eta)\xi^m = -\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m$ is true.
Then

$$
\begin{aligned}
L^i_{hj}(x,\rho\xi,\sigma\eta) &= \sigma\rho^{-1}\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m + \rho\sigma^{-1}\frac{\partial F^i_{hm}}{\partial y^j}(x,\eta)\xi^m \\
&= (\sigma\rho^{-1} - \rho\sigma^{-1})\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m
\end{aligned} \tag{3.4}
$$

is satisfied. Therefore

$$
\begin{aligned}
&L^i_{hj}(x,\rho\xi,\sigma\eta)(\rho\xi^j - \sigma\eta^j) \\
&= (\sigma\rho^{-1} - \rho\sigma^{-1})\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m(\rho\xi^j - \sigma\eta^j) \\
&= (\sigma\rho^{-1} - \rho\sigma^{-1})(\rho\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m\xi^j - \sigma\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m\eta^j) \\
&= (\sigma\rho^{-1} - \rho\sigma^{-1})(\rho\frac{\partial F^i_{hm}}{\partial y^j}(x,\xi)\eta^m\xi^j + \sigma\frac{\partial F^i_{hm}}{\partial y^j}(x,\eta)\xi^m\eta^j) = 0
\end{aligned} \tag{3.5}
$$

Then we have a following theorem.

Theorem 3.1 *For a point $x = (x^i)$ on M and a pair (ξ,η) of directions at x, if $W^i_{hj}(x,\xi,\eta) = 0,\ L^i_{hj}(x,\xi,\eta) = 0,\ K^i_{hjk}(x,\xi,\eta) = 0$ are satisfied, then following equations are satisfied for any positive number $\rho > 0, \sigma > 0$.*
(3.6)
$W^i_{hj}(x,\rho\xi,\sigma\eta)(\rho\xi^j + \sigma\eta^j) = 0,\ L^i_{hj}(x,\rho\xi,\sigma\eta)(\rho\xi^j - \sigma\eta^j) = 0,\ K^i_{hjk}(x,\rho\xi,\sigma\eta)\sigma\eta^j\rho\xi^k = 0.$

Remark 3.1 *At a point x and a pair (ξ,η) of directions satisfying the conditions of Theorem* 2.1*, then*

$$\bar{V}_r = V_r \tag{3.7}$$

is satisfied in the sense of approximate of second order with respect to ξ,η.

Next we also have following three propositions.

Proposition 3.1 *Let $F\Gamma = (N^i_j(x,y), f^i_{hj}(x,y), C^i_{hj}(x,y))$ be a Finsler connection satisfying $T^i_{rj} = 0$ and $D^i_j = 0$. Then $W = 0$ on $\mathcal{H}$ is equivalent to $L = 0$ on $\mathcal{H}$.*

Proof
From $W^i_{hj}(x,y,z) = 0$, $F^i_{hj}(x,y) = F^i_{hj}(x,z)$ is satisfied. This implies

$$\frac{\partial F^i_{hj}}{\partial y^k}(x,y) = \frac{\partial F^i_{hj}}{\partial z^k}(x,z) = 0. \tag{3.8}$$

Therefore $L^i_{hj}(x, y, z) = 0$ is satisfied.
Inversely, we assume $L^i_{hj}(x, y, z) = 0$. Then the following equation

$$\frac{\partial F^i_{hm}}{\partial y^j}(x, y)z^m = -\frac{\partial F^i_{hm}}{\partial z^j}(x, z)y^m \tag{3.9}$$

is satisfied on any point (x, y, z). We take partial derivations by y^l and z^k of both sides, respectively. Then we have

$$\frac{\partial^2 F^i_{hk}}{\partial y^j \partial y^l}(x, y) = -\frac{\partial^2 F^i_{hl}}{\partial z^j \partial z^k}(x, z). \tag{3.10}$$

This means that the derivative of the secon order by the second variable of the coefficient F^i_{hj} has no the second variable. Namely,

$$\frac{\partial F^i_{hj}}{\partial y^k}(x, y) = Q^i_{hjkm}(x)y^m \tag{3.11}$$

is satisfied.

On the other hand, $F^i_{hj}(x, y)$ is positively homogeneous of degree 0 with respect to the variable y. So $\frac{\partial F^i_{hj}}{\partial y^k}(x, y)y^k = 0$ is satisfied. Therefore we have

$$Q^i_{hjkm}(x)y^m y^k = 0. \tag{3.12}$$

The above quadratic form of y is satisfied on any y, so $Q^i_{hjkm}(x) = 0$ must be true. Therefore we have $\frac{\partial F^i_{hj}}{\partial y^k}(x, y) = 0$. Namely,

$$W^i_{hj}(x, y, z) = 0 \tag{3.13}$$

is satisfied.

Q.E.D.

In addition, according to the above proof, we have a following proposition.

Proposition 3.2 *Let $F\Gamma = (N^i_j(x, y), f^i_{hj}(x, y), C^i_{hj}(x, y))$ be a Finsler connection satisfying $T^i_{rj} = 0$ and $D^i_j = 0$. If $W = 0$ is satisfied on $\mathcal{H}$, then $\frac{\partial F^i_{hj}}{\partial y^k}(x, y) = 0$, namely, $F^i_{hj} = F^i_{hk}(x)$ is satisfied on TM.*

Further, if we assume $W^i_{hj}(x, y, z) = 0$ and $K^i_{hjk}(x, y, z) = 0$ on $\mathcal{H}$, then we can prove a following proposition.

Proposition 3.3 *Let $F\Gamma = (N^i_j(x, y), f^i_{hj}(x, y), C^i_{hj}(x, y))$ be a Finsler connection satisfying $T^i_{rj} = 0$ and $D^i_j = 0$. If, for any point (x, y, z) on $\mathcal{H}$, $W^i_{hj}(x, y, z) = 0$ and $K^i_{hjk}(x, y, z) = 0$ are satisfied, then the torsion tensor fields $P^i_{hj}(x, y), R^i_{hj}(x, y), C^i_{hj}(x, y)$ and the curvature tensor fields $R^i_{hjk}(x, y), P^i_{hjk}(x, y)$ of $F\Gamma$ satisfy following equations:*

$$P^i_{hj}(x, y) = 0,\ R^i_{hj}(x, y) = 0,\ P^i_{hjk}(x, y) + C^i_{hk|j}(x, y) = 0,\ R^i_{hjk}(x, y) = 0. \tag{3.14}$$

Proof

Since Proposition 3.2, $\frac{\partial F^i_{hj}}{\partial y^k}(x,y) = 0$ is satisfied. From $P^i_{hj} = \frac{\partial N^i_h}{\partial y^j} - F^i_{jh}$ and $N^i_j = y^m F^i_{mj}(D=0)$,

$$P^i_{hj}(x,y) = 0 \tag{3.15}$$

is satisfied. Next, from $\frac{\partial F^i_{hj}}{\partial y^k} = P^i_{hjk} + C^i_{hk|j} - C^i_{hq}P^q_{jk}$ and $P^i_{hj} = 0$, we have

$$P^i_{hjk}(x,y) + C^i_{hk|j}(x,y) = 0. \tag{3.16}$$

And from $K^i_{hjk}(x,y,z) = 0$, of course $K^i_{hjk}(x,y,y) = 0$ is satisfied. In addition, in general, $K^i_{hjk}(x,y,y) = R^i_{hjk}(x,y) - C^i_{hm}(x,y)R^m_{jk}(x,y)$ is true. So $R^i_{hjk} - C^i_{hm}R^m_{jk} = 0$ is satisfied. And from $R^i_{jk} = y^m(R^i_{mjk} - C^i_{ms}R^s_{jk})$, we have

$$R^i_{jk}(x,y) = 0. \tag{3.17}$$

We put the above conclusion in $R^i_{hjk} - C^i_{hm}R^m_{jk} = 0$ again and we have

$$R^i_{hjk}(x,y) = 0. \tag{3.18}$$

Q.E.D.

Now the author stated in detail the conditions for a Finsler space to be locally Minkowski space in [7]. If we apply Proposition 3.3 to a Finsler space with the property of h-metrical, then according to [7], we have a following theorem.

Theorem 3.2 *Let (M, F) be a Finsler space with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk} = 0$, $D^i_j = 0$ and $g_{ij|k} = 0$.*
If W and K vanish on $\mathcal{H}$, then the Finsler space $(M, F, F\Gamma)$ is a locally Minkowski space and the inverse property is also true.

Remark 3.2 *From Theorem* 3.1, *Remark* 3.1 *and Theorem* 3.2, *we notice that a locally Minkowski space is "flat in the sense of linear parallel displacement".*

Last, we state three lemmas that can be proved from the definition, easily.

Lemma 3.2 *For $W^i_{hj}(x,y,z) = F^i_{hj}(x,y) - F^i_{hj}(x,z)$, following equations are satisfied.*

$$W^i_{hj}(x,z,y) = -W^i_{hj}(x,y,z) \tag{3.19}$$

$$W^i_{hj}(x,y,y) = 0 \tag{3.20}$$

$$W^i_{hj}(x,y,z) + W^i_{hj}(x,-z,y) + W^i_{hj}(x,-y,-z) + W^i_{hj}(x,z,-y) = 0 \tag{3.21}$$

Lemma 3.3 *For $L^i_{hj}(x,y,z) = \frac{\partial F^i_{hm}}{\partial y^j}(x,y)z^m + \frac{\partial F^i_{hm}}{\partial z^j}(x,z)y^m$, following equations are satisfied.*

$$L^i_{hj}(x,z,y) = L^i_{hj}(x,y,z) \tag{3.22}$$

$$L^i_{hj}(x,y,y) = 2\frac{\partial F^i_{hm}}{\partial y^j}(x,y)y^m = 2(P^i_{h0j} + C^i_{hj|0} - C^i_{hm}P^m_{0j}) \tag{3.23}$$

$$L^i_{hj}(x,y,z) + L^i_{hj}(x,-z,y) + L^i_{hj}(x,-y,-z) + L^i_{hj}(x,z,-y) = 0 \tag{3.24}$$

Lemma 3.4 *For* $K^i_{hjk}(x,y,z) = \frac{\delta F^i_{hj}}{\delta x^k}(x,y) - \frac{\delta F^i_{hk}}{\delta x^j}(x,z) - F^i_{mj}(x,y)F^m_{hk}(x,y) + F^i_{mk}(x,z)F^m_{hj}(x,z)$, *following equations are satisfied.*

$$K^i_{hjk}(x,z,y) = -K^i_{hkj}(x,y,z) \tag{3.25}$$

$$K^i_{hjk}(x,y,y) = R^i_{hjk}(x,y) - C^i_{hm}(x,y)R^m_{jk}(x,y) \tag{3.26}$$

$$K^i_{hjk}(x,y,z) - K^i_{hkj}(x,y,z) = K^i_{hjk}(x,y,y) - K^i_{hkj}(x,z,z) \tag{3.27}$$

$$K^i_{hjk}(x,y,z) + K^i_{hjk}(x,z,y) = K^i_{hjk}(x,y,y) + K^i_{hjk}(x,z,z) \tag{3.28}$$

$$K^i_{hjk}(x,-y,z) + K^i_{hjk}(x,y,-z) = K^i_{hjk}(x,y,z) + K^i_{hjk}(x,-y,-z) \tag{3.29}$$

$$\begin{aligned} &K^i_{hjk}(x,y,z) + K^i_{hjk}(x,-z,y) + K^i_{hjk}(x,-y,-z) + K^i_{hjk}(x,z,-y) \\ &= K^i_{hjk}(x,y,y) + K^i_{hjk}(x,-y,-y) + K^i_{hjk}(x,z,z) + K^i_{hjk}(x,-z,-z) \end{aligned} \tag{3.30}$$

Remark 3.3 *If* F^i_{hj} *is the coefficient of Riemannian connection, then* $W^i_{hj} \equiv 0$, $L^i_{hj} \equiv 0$ *are satisfied and* K^i_{hjk} *is the coefficient of Riemannian curvature.*

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